Abstract

With the aim of exploring the impact of continuous measurement in quantum gravity, the paper delves into various Sachdev-Ye-Kitaev (SYK) models. The SYK model is a quantum mechanical system that consists of a large number of Majorana fermions and exhibits the properties of a one-dimensional conformal field theory in the large N and low-energy limits, which is dual to two-dimensional quantum gravity. We first studied the low-energy effective action of the Maldacena & Qi model (MQ model), namely the Schwarzian action, and explored its dynamics as well as its relationship with wormholes. In the higher-energy region of this model, the dynamics are dominated by the Schwinger-Dyson (SD) equations. These techniques will be applied in the study of measurement-induced phase transitions (MIPT) that appear in the SYK model and its gravitational dual, as well as in the Lindbladian SYK model of open systems. We reviewed the detailed derivations of these theories and supplemented key details on both the Jackiw-Teitelboim (JT) gravity side and the SYK side, thereby deepening our understanding of the nature of the JT/SYK duality and the broad applications of the MQ model.

Key words: Quantum gravity, Holographic principle, Jackiw-Teitelboim gravity, Sachdev-Ye-Kitaev model, Wormhole, Measurement-induced phase transition, Lindbladian dynamics

Contents

Chapter 1 Introduction

Motivation

Measurement plays a crucial role in quantum mechanics. Through the holographic principle, we gain insights into how duality can connect two systems that appear vastly different at first glance. This allows us to investigate how measurements affect quantum gravity, a question that many people care about deeply and which serves as the motivation for the research presented in this paper. In this project, we aim to explore the implications of measurements in gravitational systems through our study of the Sachdev-Ye-Kitaev (SYK) model.

Holography:

In the 1970s, the discovery that black hole entropy is proportional to the area of its event horizon rather than its volume led physicists to realize that the true dynamical degrees of freedom in a gravitational system may correspond to physics existing in one fewer dimension. This insight gave rise to the holographic principle, which later evolved into the AdS/CFT correspondence conjecture. There is a duality between $n+1$ dimensional Anti-de Sitter gravity and n-dimensional Conformal Field Theory. In 1997, Maldacena provided a top-down realization of this correspondence for AdS_5/CFT_4.[1][2]. And more specific version of general AdS/CFT refers to GKPW dictionary[3]

$$Z_{\text{grav}}[\phi_0^i(x);\partial M] = \left\langle \exp\left(-\sum_i \int d^d x \phi_0^i(x) O^i(x)\right)\right\rangle_{\text{CFT on }\partial M}$$

JT SYK:

Currently, one of the worldwide research interest lies in the study of the AdS_2/CFT_1,particularly on the duality between Jackiw- Teitelboim Gravity and Sachdev-Ye-Kitaev model. JT/SYK dual JT gravity is one of the 2D dilaton gravity model. Free JT gravity has no dynamic in its bulk in pure 2D Einstein gravity. All the dynamic in JT gravity lies on its nearly AdS_2 boundary. The effective description for this low energy region would be suit in Schwarzian Action which is the simplest action preserving $SL(2,R)$ symmetry. SYK side would ultimately attain similar description and the similar effective description implies those two system has consistent behavior. SYK is a model lives in 0+1-dimensional, where there are well-defined quantum mechanics.[4][5].

Holographic dictionary helps us map onto gravitational systems, enabling us to discuss concepts like entanglement in gravitational system. In certain parameter range, there would be a gravitational description as wormholes[6][7][8]. And manipulation on quantum system would help us gain vision on wormhole system as what Google did in 2019[9]. Study measurement using JT/SYK dual would delve deeper in our the understanding of gravity, and would also have the chance to bridge the gap between formal theory and laboratory experiments, and may in turn deepen our comprehension of measurement.

Measurement:

As for measurement, with the flourishing of quantum computing and AMO physics, a deeper understanding of measurement has become increasingly significant. Axioms of measurement tells that after a measurement, the wavefunction collapse to one eigenstate of the operator represented the observable being measured.We will introduce a description of measurements in field theory systems. Generally, for non-equilibrium systems, the Lindbladian dynamics described in the Schwinger-Keldysh Path Integral formalism will be used[10][8], and in the article we focus on[11], the weak projection approximation is used to obtain a dynamical description very similar to that in Maldacena and Qi[6]. Additionally, Milekhin's work mentions the observation of Measurement Induced Phase Transition phenomena. In the model they discussed, the well-defined measurement-induced phase transition (MIPT) on the field theory side can be given by holography on the gravitational side as traversable wormholes and the phase transition of two black holes, which is of great significance for studying measurements on the gravitational side.

Arrangement Of The Thesis

The holographic principle, the JT/SYK duality, and MIPT are introduced so that readers can understand the motivation of this research. Due to time constraints and my limited academic ability, I am unable to conduct more in-depth research. This paper mainly presents the important conclusions and corresponding derivations of the JT/SYK duality to MIPT. The structure of this paper is as follows. In Chapter 2, the basic setup of the SYK model and the corresponding techniques are introduced. In Chapter 3, the duality between JT gravity and SYK is discussed in the context of the Schwarzian region, which is extremely important for the subsequent discussion of the low-energy dynamics of the MQ model and other models. In Chapter 4, the construction of the MQ model in the large N low-energy limit is discussed, and in Chapter 5, some techniques for solving the MQ model are presented. After the basic discussion, in Chapter 6, some models different from the MQ model are discussed, which can describe measurement protocols in specific situations. Finally, we further discuss topics beyond measurement. In Chapter 7, we briefly discuss the description of SYK in open systems, the corresponding Lindbladian dynamics, and their solutions.

It is evident that completing all the above discussions is a massive project. Therefore, as a mediocre undergraduate thesis, this paper cannot possibly cover every aspect, and not all key steps can be explained thoroughly. The strategy adopted in this paper is to mark the content that cannot be successfully reproduced with • and directly cite the conclusions from the relevant papers.

Chapter 2 SYK Model

2.1 The Model

2.1.1 Physical Meaning of SYK Model

The following physics intuition is referred to the introduction given by Prof. Zhang at the South East Univ summer camp.[12]

The Heisenberg model describes the coupling of neighboring spins with a fixed coupling constant, given by the Hamiltonian $H = \sum_{\langle x,y\rangle} J\hat{S}_x\hat{S}_y$.

Sachdev and Ye upgraded this model to $H = \sum_{i<j} J_{ij}\hat{S}_i\hat{S}_j$, which can be extended to $SU(2)$, $SU(N)$, and further generalized to the $SU(N)$ 4-fermion model.[13]

Kitaev simplified the model by focusing on the coupling of Majorana fermions, describing N quantum dots where any q dots are coupled, and the coupling is a Gaussian random variable.

2.1.2 Mathematical Settings of SYK Model

The Hamiltonian of the SYK model is

$$\boxed{H = i^{\frac{q}{2}} \sum_{1\le i_1<\ldots<i_q\le N} J_{i_1\ldots i_q}\psi_{i_1}\ldots\psi_{i_q}} \tag{2-1}$$

The SYK model is characterized by the following properties:

1. The fields ψ_i are Majorana fermions, which satisfy the conditions

$$\psi^\dagger = \psi \quad \text{and} \quad \{\psi_i, \psi_j\} = \delta_{ij}, \quad i, j = 1, \ldots, N. \tag{2-2}$$

2. The parameter q is restricted to even integers, i.e., $q \in 2\mathbb{Z}$. This ensures that the Hamiltonian is invariant under the exchange of fermionic indices.
3. The factor $i^{\frac{q}{2}}$ is introduced to ensure the Hermiticity of the Hamiltonian. Specifically, it guarantees that

$$H^\dagger = (-i)^{\frac{q}{2}} J_{i_1\ldots i_q}\psi_{i_q}\ldots\psi_{i_1} = i^{\frac{q}{2}} J_{i_1\ldots i_q}\psi_{i_1}\ldots\psi_{i_q} = H. \tag{2-3}$$

4. The couplings $J_{i_1\ldots i_q}$ are real random Gaussian variables with the following properties:

$$\langle J_{i_1\ldots i_q}\rangle = 0, \quad \langle J^2_{i_1\ldots i_q}\rangle = \sigma^2, \tag{2-4}$$

where the variance σ is given by

$$\sigma = \sqrt{(q-1)!}\frac{J}{N^{\frac{q-1}{2}}}. \tag{2-5}$$

The Gaussian distribution of these couplings has a weight factor of

$$e^{\frac{J^2_{i_1\ldots i_q}}{2\sigma^2}}. \tag{2-6}$$

5. For the specific case of $q = 4$, the Hamiltonian is often expressed as

$$H = \sum_{ijkl=1}^{N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l. \tag{2-7}$$

This form highlights the quartic interaction among the Majorana fermions with random coupling.J_{ijkl} may seem to take arbitrary values, but we must not forget $\psi_i\psi_j\psi_k\psi_l$ are fermions following the anti-commutation law. Therefore the random variable J_{ijkl} must be antisymmetric:

$$J_{ijkl} = -J_{ijlk} \tag{2-8}$$

2.2 Schwinger Dyson Equation and $G\Sigma$ Action

Once we have the expression of green function with self-eregy correction of a theory, a model can be "Solved". We can draw the feynmann diagram to express the relation of self energy and green funciton. In references[4],[14],[15], and numerous related literature, there is a detailed discussion of the Feynman rules for the SYK model. The Green's function G^{free} and the associated self-energy corrections are calculated accordingly. For the SYK$_q$ model, the Schwinger-Dyson equation shows in the following

$$G = [\partial_\tau - \Sigma]^{-1}, \tag{2-9}$$
$$\Sigma(\tau,\tau') = J^2\,[G(\tau,\tau')]^{q-1}. \tag{2-10}$$

However, it would be hard to discuss the theory all the way through feymann diagram. In this section, we focus on describing the SYK model using the path integral approach. By introducing the effective fields G and Σ . They are no longer be seen as green function and self energy function but fields in current context, when those fields becomes on-shell, their

EOM describes the correct Schwinger Dyson equations. Therefore, we can study the SYK model by examining the new $G\Sigma$ action together with the on-shell conditions.

This section first discusses how to take the ensemble average of the original action, that is, integrating over the random coupling. Then, I'll discusses the $G\Sigma$ action for Majorana fermions and derives the SD equations for this action. Some steps may seem arbitrary, and here we will also compare the $G\Sigma$ actions for real scalars and complex scalars to deepen the understanding of the method we use in deriving $G\Sigma$ action.

2.2.1 Ensemble average over random variables

In general case, we should perform ensemble averaging on observable physical quantities, i.e., $\langle \ln Z \rangle$, but this calculation is quite complex. On the other hand, it can be shown that in the large N limit, $\langle \ln Z \rangle \approx \ln\langle Z \rangle$, so we directly perform ensemble averaging on the partition function. Therefore the partition function after ensemble average would be given by

$$\langle Z \rangle = \int \mathcal{D}\psi \int dJ_{\dots} e^{-\frac{J_{\dots}^2}{2\sigma^2}} e^{-I}, \tag{2-11}$$

where I is the Euclidean space action represented as

$$I = \frac{1}{2}\psi_i \partial \psi^i - H. \tag{2-12}$$

It is known that the gaussian integration gives us

$$\int dJ e^{-\frac{J^2}{2\sigma^2} - JX} = e^{\frac{\sigma^2 X^2}{2}}.$$

The function we need to perform Gaussian integral averaging is as follows:

$$\exp\left[- \int i^{\frac{q}{2}} J_{i_1 \cdots i_q} \psi^{i_1} \cdots \psi^{i_q} \right].$$

Since $\langle J_{\dots} \rangle = 0$ and it is a random variable, there is index pairing, which is not going to be detailedly discussed here. Finally, we obtain the result after ensemble averaging:

$$\langle Z \rangle_J \sim \prod_i \int D\psi_i \exp\left(- \int d\tau \frac{1}{2} \sum_i \psi_i \partial_\tau \psi_i \right. \tag{2-13}$$

$$\left. + \sum_{1 \le i_1 < \cdots < i_q \le N} i^q \frac{(q-1)! J^2}{2N^{q-1}} \int \int d\tau d\tau' (\psi_{i_1} \cdots \psi_{i_q})(\tau)(\psi_{i_1} \cdots \psi_{i_q})(\tau') \right). \tag{2-14}$$

The contribution of the interaction term after ensemble averaging is as follows,

$$\exp\left[i^q \frac{(q-1)!J^2}{2N^{(q-1)}}\left(\int d\tau \psi^{i_1}\cdots\psi^{i_q}\right)^2\right]. \tag{2-15}$$

Calculation for $q = 4$

To make the calculation method clearer, we take $q = 4$ for a simple calculation (no need to consider the i factor here):

$$\sum_{1\le j<k<l\le N}(\psi_i\psi_j\psi_k\psi_l)(\tau)(\psi_i\psi_j\psi_k\psi_l)(\tau') = \frac{1}{4!}\left[\sum_i \psi_i(\tau)\psi_i(\tau')\right]^4. \tag{2-16}$$

Proof

$$\text{lhs} = \sum_{1\le i<j<k<l\le N}\psi_i\psi_j\psi_k\psi_l(\tau)\psi_i\psi_j\psi_k\psi_l(\tau') = \frac{1}{4!}\sum_{i\ne j\ne k\ne l}\psi_i\psi_j\psi_k\psi_l(\tau)\psi_i\psi_j\psi_k\psi_l(\tau'). \tag{2-17}$$

This step is valid due to the combinatorial property (also know as convert ordered summation to unordered summation)

$$\sum_{1\le i<j<k<l\le N} = \frac{1}{4!}\sum_{i\ne j\ne k\ne l}. \tag{2-18}$$

Note that $\psi\psi'$ has already been time-ordered, i.e., $\psi_a(\tau)\psi_b(\tau')$, so we only need to group the same indices together. By group theory, we know that the parity of the number of swaps needed for this rearrangement is the same, so without loss of generality, we start rearranging from the l index. This process requires $3+2+1$ adjacent swaps, which does not contribute to the sign difference. Thus, we obtain the following result:

$$\begin{aligned}\frac{1}{4!}\sum_{i\ne j\ne k\ne l}\psi_i\psi_j\psi_k\psi_l\psi_i'\psi_j'\psi_k'\psi_l' &= \frac{1}{4!}\sum_{i\ne j\ne k\ne l}\psi_i\psi_i'\psi_j\psi_j'\psi_k\psi_k'\psi_l\psi_l' \\ &= \frac{1}{4!}\left(\sum_i\psi_i\psi_i'\right)\left(\sum_j\psi_j\psi_j'\right)\left(\sum_k\psi_k\psi_k'\right)\left(\sum_l\psi_l\psi_l'\right)_{i\ne j\ne k\ne l}.\end{aligned}$$

Since $\psi^2(\tau) = 0$, we have

$$\sum_a\psi^a\times\sum_b\psi^b = \sum_{a\ne b}\psi^a\psi^b, \tag{2-19}$$

so we can drop the restriction $i \neq j \neq k \neq l$, and thus obtain rhs. QED.

Remark:

There is a subtle point regarding the Gaussian integral and resummation. Here, we follow the procedure of first performing the ensemble average over the random variable $J_{\cdot}$, and then converting the ordered summation of Gaussian variables into an unordered summation, i.e., $\sum_{1\le i_1\le\cdots\le i_q\le N} \Rightarrow \frac{1}{q!}\sum_{i_1\neq\cdots\neq i_q}$. While it is theoretically possible to interchange the order, it is better in practice to first perform the ensemble average over the random variable and then change the summation order. Otherwise, the following paradox may arise:

Consider the classic SYK$_4$ model. If we first perform the unordered summation, the object we need to average over the ensemble is:

$$\exp\left[-\int \frac{1}{4!} J_{ijkl}\psi^i\psi^j\psi^k\psi^l\right], \tag{2-20}$$

Given the integral formula $\int dJ_{ijkl} e^{-\frac{J_{ijkl}^2}{2\sigma^2} - J_{ijkl}X} = e^{\frac{\sigma^2 X^2}{2}}$, we obtain the interaction part after ensemble averaging as follows, which seems to lead to a paradox:

$$\frac{3J^2}{N^3}\cdot\frac{1}{(4!)^2}\left(\left(\sum_{i_1\neq\cdots\neq i_q}\int dt\,\psi^i\psi^j\psi^k\psi^l\right)\times\left(\sum_{i_1\neq\cdots\neq i_q}\int dt'\,\psi'^i\psi'^j\psi'^k\psi'^l\right)\right) \tag{2-21}$$

Note that in our previous discussion, the summation we described was $\sum_{1\le i_1\le\cdots\le i_q\le N}$, whereas the summation we are discussing now is $\sum_{i_1\neq\cdots\neq i_q}$. Compared to the previous form, this allows i, j, k, l to be permuted, resulting in a contribution of $q!$ and we have to count in this symmetry contribution.

Generalization to Arbitrary q

Note that the above discussion only holds for $q = 4$. To generalize to arbitrary q coupling, we need $\frac{q^2-q}{2}$ adjacent swaps, which may contribute to the sign factor. Bringing back the original i factor, we can see that the signs exactly cancel out.

$$
\begin{aligned}
\text{lhs} &\propto i^q \sum_{1\le i_1<i_2<\cdots<i_q\le N} \psi_{i_1}\psi_{i_2}\cdots\psi_{i_q}(\tau)\psi_{i_1}\psi_{i_2}\cdots\psi_{i_q}(\tau') \\
&= i^q \frac{1}{q!}\sum_{i_1\neq i_2\neq\cdots\neq i_q} \psi_{i_1}\psi_{i_2}\cdots\psi_{i_q}(\tau)\psi_{i_1}\psi_{i_2}\cdots\psi_{i_q}(\tau') \\
&= (-1)^{\frac{q^2-q}{2}}\cdot(-1)^{\frac{q}{2}}\frac{1}{q!}\sum_{i_1\neq i_2\neq\cdots\neq i_q} \psi_{i_1}\psi'_{i_1}\psi_{i_2}\psi'_{i_2}\cdots\psi_{i_q}\psi'_{i_q} \\
&= \frac{1}{q!}\left(\sum_{i_1}\psi_{i_1}\psi'_{i_1}\right)\left(\sum_{i_2}\psi_{i_2}\psi'_{i_2}\right)\cdots\left(\sum_{i_q}\psi_{i_q}\psi'_{i_q}\right)_{i_1\neq i_2\neq\cdots\neq i_q}.
\end{aligned}
$$

2.2.2 $G\Sigma$ Actions and SD eqns For Majorana Fermions

Majorana fermions are the case we concerned in following pages, though different field can be used with corresponding modification. Now we are going to deal the integration over ψ to derive the functional determinant at first. Then we are going to derive the $G\Sigma$ action as follows. Doing the variation over G, Σ would give the SD eqns in the end.

2.2.2.1 Functional Determinant:

Sarosi mentions the use of Gaussian-Berezin-integral:

$$
\int \mathcal{D}\psi e^{-\frac{1}{2}\psi\cdot A\cdot\psi} = \text{Constant}\cdot\sqrt{\det A} \tag{2-22}
$$

Many references, such as[16], write the right-hand side as related to $\text{Pf}(A)$, with the definition as follows:

$$
e^{\frac{1}{2}\text{Tr}[\log(A)]} = \text{Pf}(A) \tag{2-23}
$$

For more properties of Grassmann number integrals, see[17].

Note that for infinite-dimensional linear spaces, the continuous version of matrix dot product corresponds to

$$
\int d\tau_1 d\tau_2\, \psi(\tau_1)A(\tau_1,\tau_2)\psi(\tau_2) \tag{2-24}
$$

which is why we need to discuss with integral parameters without getting tired. And some definitions that may be used are as follows.

2.2.2.2 Inserting $G\Sigma$ Identity:

After performing the ensemble average of $J_{\dots}$ in the SYK model in last section, we obtain the effective action in the following form:

$$I = \int d\tau \frac{1}{2}\psi^i(\tau)\partial_\tau\psi_i(\tau) + \text{Constant} \cdot \int d\tau_1 d\tau_2 \bullet \left(\sum_{i=1}^{N} \frac{1}{N}\psi_i(\tau_1)\psi_i(\tau_2)\right)^q \tag{2-25}$$

To conveniently write it in the form of a matrix quadratic form later, we rewrite $\int d\tau\psi(\tau)\partial_\tau\psi(\tau)$ as

$$\int d\tau_1 d\tau_2 \psi(\tau_1) \cdot \delta(\tau_{12})\partial_{\tau_1}\psi(\tau_2) \tag{2-26}$$

$G\Sigma$ Identity: Insert it at the position of the red dot above

$$\begin{aligned}
1 &\sim \delta\left(G(\tau_1,\tau_2) - \frac{1}{N}\sum_{i=1}^{N}\psi_i(\tau_1)\psi_i(\tau_2)\right) \\
&\propto \int \mathcal{D}\Sigma(\tau_1,\tau_2)\exp\left(-\frac{N}{2}\Sigma(\tau_1,\tau_2)\left(G(\tau_1,\tau_2) - \frac{1}{N}\sum_{i=1}^{N}\psi_i(\tau_1)\psi_i(\tau_2)\right)\right)
\end{aligned}$$

Here we rewrite the integral in the form of an inner product $\cdot$, so the ψ part in the partition function can be deformed as follows (with $\tau_{1,2}$ representing the row and column indices of the operator respectively):

$$\begin{aligned}
&\prod_i \int \mathcal{D}\psi_i \exp\left[-\sum_i \frac{1}{2}\psi_i \cdot \delta(\tau_{12})\partial_{\tau_1} \cdot \psi_i + \frac{1}{2}\sum_i \psi_i \cdot \Sigma \cdot \psi_i\right] \\
=&\prod_i \left(\int \mathcal{D}\psi_i e^{-\frac{1}{2}\psi_i\cdot(\delta(\tau_{12})\partial_{\tau_1}-\Sigma)\cdot\psi_i}\right) \\
\propto&\prod_i \det(\delta(\tau_{12})\partial_{\tau_1} - \Sigma)^{\frac{1}{2}} \\
=&e^{\frac{N}{2}\ln\det(\delta(\tau_{12})\partial_{\tau_1}-\Sigma)} \\
=&e^{\frac{N}{2}\mathrm{Tr}\ln(\delta(\tau_1,\tau_2)\partial_\tau-\Sigma)}
\end{aligned}$$

The last step uses the equality $\mathrm{Tr}\ln M = \ln\det M$, and under the consideration of Tr, we no longer need to consider $\delta(\tau_{12})$, and it can also be ignored in $\ln\det$.

Considering $Z(J) \sim \int DGD\Sigma e^{-NI[G,\Sigma]}$, then we can find

$$-\frac{1}{2}\mathrm{Tr}\ln(\delta(\tau_1,\tau_2)\partial_{\tau_2} - \Sigma) \subset I \tag{2-27}$$

2.2.2.3 Deriving the First Part of the SD Equation:

The first part is the variation of Σ,

$$I \supset -\frac{1}{2}\mathrm{Tr}\ln(\delta(\tau_1,\tau_2)\partial_{\tau_2}-\Sigma)+\frac{1}{2}\iint d\tau_1 d\tau_2 \Sigma(\tau_1,\tau_2)G^{\psi}(\tau_1,\tau_2) \tag{2-28}$$

Varying Σ gives

$$\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\frac{-\delta\Sigma}{\delta(\tau_1,\tau_2)\partial_\tau-\Sigma}+\frac{1}{2}\iint d\tau_1 d\tau_2 \delta\Sigma(\tau_1,\tau_2)G^{\psi}(\tau_1,\tau_2)\\
=&\frac{1}{2}\mathrm{Tr}(\delta\Sigma\cdot(\delta(\tau_1,\tau_2)\partial_\tau-\Sigma)^{-1})+\cdots\\
=&\frac{1}{2}\mathrm{Tr}(\int d\tau_2\delta\Sigma(\tau,\tau_2)\cdot(\delta(\tau_1,\tau_2)\partial_\tau-\Sigma)^{-1}(\tau_2,\tau'))+\cdots\\
=&\frac{1}{2}\iint d\tau_1 d\tau_2\ \delta\Sigma(\tau_1,\tau_2)\cdot\left((\delta(\tau_1,\tau_2)\partial_\tau-\Sigma)^{-1}(\tau_2,\tau_1)+G^{\psi}(\tau_1,\tau_2)\right)
\end{aligned}$$

Due to the nature of fermions, considering $G^{\psi}(\tau_1,\tau_2)=-G^{\psi}(\tau_2,\tau_1)$(with detailed discussion in appendix B), after swapping the order, we can regard the operator $(\delta(\tau_1,\tau_2)\partial_\tau-\Sigma)^{-1}\cong G^{\psi}$, so we have $\delta(\tau_1,\tau_2)\partial_\tau G-\Sigma\cdot G=\mathbf{1}$.

When we assume that G has translational symmetry (which can be given by the second SD equation in the classical SYKq model), we can write $\Sigma\cdot G$ in the form of a convolution, thus allowing us to solve it better in Fourier space.

In the process of derivation above, there are some elements that may seem arbitrary, such as the $\frac{N}{2}$ in the exponential when inserting the identity matrix $\mathbf{1}$, as well as some derivations involving functional determinants. These aspects will not change within the scope of the MQ models we are discussing. However, for further research into different field types of MQ models, a more in-depth discussion of the standard techniques of the Sachdev-Ye-Kitaev model is provided in appendix A.

Chapter 3 JT & SYK

We have understood that the SYK model exhibits properties of a one-dimensional conformal field theory in the large-N and the low-energy limit. We refer to this parameter section as the Conformal Region. The subject of this chapter is to discuss what happens when we move away from this region. We will find some thing very interesting. The effective action of the SYK model slightly above conformal region and the effective action of JT gravity near AdS_2 boundary have the same form, which can be seen as a realization for the AdS_2/CFT_1 correspondence, as if there is a CFT_1 living on the boundary of AdS_2 gravity system.

In this chapter, we will introduce the Schwarzian dynamics of the SYK model and the Schwarzian dynamics of JT Gravity. We will then discuss the dynamical properties in detail. Specifically, we will first discuss the mathematical properties of the Schwarzian itself and then discuss the Equation of Motion (EOM) of the Schwarzian Action.

3.1 SYK Model

3.1.1 Go Beyond Conformal Region

In our previous discussion, we have obtained the $G - \Sigma$ action. Below, $\tilde{G}, \tilde{\Sigma}$ correspond to the effective fields that are off-shell when considering the path integral $\int \mathcal{D}G\Sigma e^{-NI}$.

$$I \equiv \frac{S}{N} = -\frac{1}{2}\log\det(\partial_t - \widetilde{\Sigma}) + \frac{1}{2}\int d\tau_1 d\tau_2 \left[\widetilde{\Sigma}(\tau_1,\tau_2)\widetilde{G}(\tau_1,\tau_2) - \frac{J^2}{q}\widetilde{G}(\tau_1,\tau_2)^q\right] \tag{3-1}$$

Also, because they are not on shell, we have the freedom to manipulate the Σ field. We choose to translate $\Sigma \longrightarrow \Sigma + \sigma$, thus obtaining two parts $I = I_{CFT} + I_S$.

$$\frac{I_{CFT}}{N} = -\frac{1}{2}\log\det(-\tilde{\Sigma}) + \frac{1}{2}\int d\tau_1 d\tau_2 \left(\tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) - \frac{J^2}{q}\tilde{G}(\tau_1,\tau_2)^q\right) \tag{3-2}$$

$$\frac{I_S}{N} = \frac{1}{2}\int d\tau_1 d\tau_2\, \sigma(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) \tag{3-3}$$

With $\sigma(\tau_1,\tau_2) \equiv \delta(\tau_1-\tau_2)\partial_\tau$. We can easily derive the Schwinger-Dyson equations by varying I_{CFT}, and the resulting description is precisely the SYK$_q$ results for the Conformal

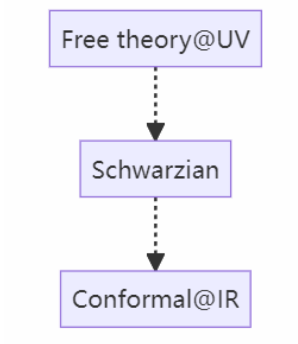

Figure 3-1 Solutions of SYK at different energy scale

Region that we discussed earlier. Or we can see in the low erergy region where $|J\tau| \gg 1$, I_S can be thrown away,since $\sigma \sim \frac{1}{\tau}$ is heavily suppressed. Therefore, We can consider the dynamics beyond the Conformal Region by study I_S.

3.1.2 Study I_S

It would be hard to discuss the behavior of σ with out any approximation. We are going to thrown it away and treate what remain in I_S perturbatively. Before explain in detail and solving the dynamics, lets firstly see the solutions of SYK at different E scale, in figure 3-1. After that we can see why it is reasonable to treat I_S as perturbation from I_{CFT}

High Energy: Through dimension analysis, we can see that $[J] = [E]$ and this means that interactive hamiltonian would be negligible at high energy scale and it becomes to a free fermion theory already solved in QFT.

Low Energy: In low energy region, we already see that the system behaves as a CFT_1 in large N limit. We solved SD eqn with the ansatz of conformal correlator.

Schwarzian Limit: Now we are moving away from deep IR region. Instead of dropping I_{CFT} in the action, what we do is actually slightly deform the conformal solution and view it as a perturbation of IR action.

Technically, this means we need to modify σ. Since I_S describes time scale $\Delta\tau \sim 0$ (Energy scale in UV), we introduce a regularization scheme, described by ϵ, to characterize the overlap between I_S and I_{CFT}. In a sense, ϵ describes how far we are from the IR region in most approach(However, not here in Suzuki's approach, where he throw the $\sigma(\tau_1, \tau_2)$ and expand τ_{12} directly.). Although methods are different and no article proved the equivalence, they would all give the the schwarzian behavior once we focus on the leading behavior proportional to the lowest order of regularization parameter. We can understood it through its physical meaning.

Physically, our operation is equivalent to expanding in terms of ∂_τ, and this operation, when analyzed dimensionally, is equivalent to expanding in terms of $(\beta J)^{-1}$. Therefore, the physical interpretation of the Schwarzian Action is the behavior of the SYK model in the low-temperature limit.

However, there are different regularization schemes. The mainstream includes those by Maldacena[4] and Kitaev[16]. In Sarosi's lecture notes[14], another method is also mentioned, but this paper adopts a relatively simpler approach, referring to Suzuki[18][19]. During the research process, a doctoral thesis involving a discussion of different methods was found[20].

3.1.3 Deriving Schwarzian:

We are going to deal with

$$\int d\tau_1 d\tau_2 \sigma(\tau_1,\tau_2)\cdot G(\tau_1,\tau_2) \tag{3-4}$$

Changing the variables by defining $\tau_{12} \equiv \tau_1 - \tau_2$ and $\tau_+ \equiv \frac{\tau_1+\tau_2}{2}$. The Jacobi factor is 1. After that, we are going to write the integral with new variables. Since there is a $\sigma(\tau_1,\tau_2)$ it means we are discussing a region where $\tau_{12} \ll 1$. It is worth noticing that we are not using ϵ to measure how close our discussion is to the $\sigma(\tau_1,\tau_2)$ in I_S. This can be regarded as us discussing a deformed case of I_S, where we discard $\sigma(\tau_1,\tau_2)$ but expand τ_{12} as a small quantity, thereby approximating our discussion.

Now we are going to deal with I_S with σ thrown out and G adopting conformal ansatz as follows:

$$b\left(\frac{|f'(\tau_1)f'(\tau_2)|^{\frac{1}{2}}}{|f(\tau_1)-f(\tau_2)|}\right)^{\frac{2}{q}} \tag{3-5}$$

We are going to expand the following factor at first:

$$\frac{1}{|f(\tau_1)-f(\tau_2)|} = \frac{1}{|f'(\tau_2)||\tau_1-\tau_2|} - \frac{|f''(\tau_2)|}{2|f'(\tau_2)|^2} + \frac{|f''(\tau_2)|^2}{4|f'(\tau_2)|^3}|\tau_1-\tau_2| - \frac{|f'''(\tau_2)|}{6|f'(\tau_2)|^2}|\tau_1-\tau_2| + \cdots \tag{3-6}$$

$$|f'(\tau_1)|^{\frac{1}{2}} = |f'(\tau_2)|^{\frac{1}{2}}\left[1 + \frac{|f''(\tau_2)|}{2|f'(\tau_2)|}|\tau_1 - \tau_2| - \frac{|f''(\tau_2)|^2}{8|f'(\tau_2)|^2}|\tau_1 - \tau_2|^2 + \frac{|f'''(\tau_2)|}{4|f'(\tau_2)|}|\tau_1 - \tau_2|^2 + \cdots\right] \tag{3-7}$$

Putting them together we'll have

$$b\left[\frac{1}{|\tau_1 - \tau_2|} + \frac{1}{12}\frac{|f'''(\tau_2)|}{|f'(\tau_2)|}|\tau_1 - \tau_2| - \frac{1}{8}\frac{|f''(\tau_2)|^2}{|f'(\tau_2)|^2}|\tau_1 - \tau_2| + \cdots\right]^{\frac{2}{q}} \tag{3-8}$$

Start from $q = 2$ case

We are going to start with $q = 2$, and we will see we have a wonderful result when $q = 2$. After that, we would use an ϵ expansion scheme to describe any q where $q = \frac{2}{1-\epsilon}$. Since we are expanding in ϵ which is less than 1, it would have finite divergent term, and converge with proper renormalization.

Note that

$$b^q = \frac{1}{\pi J^2}\left(\frac{1}{2} - \frac{1}{q}\right)\tan\frac{\pi}{q} \tag{3-9}$$

(where we used the result in Sarosi's note). And for $q = 2$, we can verify

$$\lim_{\eta \to 2}\left(\frac{1}{2} - \frac{1}{2+2\eta}\tan\frac{\pi}{2+2\eta}\right) = \frac{1}{\pi} \tag{3-10}$$

and it means $b = \frac{1}{\pi J}$.

We are going to use a proper regularization scheme to remove the first term in equation (12). Meanwhile, it has nothing to do with the dynamic mode $f(\tau)$, we can treat it as the UV behavior in the vacuum and has nothing to do with the dynamics we are talking about.

We can see that the $q = 2$ effective action when we are out of IR, is as follows. (There is an overall sign which needs to be identified carefully in the future.)

$$S_{\mathrm{Sch},q=2} = -\frac{N}{24\pi J}\int dt\,\mathrm{Sch}(f(\tau), t) \tag{3-11}$$

For general q cases

Since we already solved the $q = 2$ case quite perfectly, we are going to extend to general q cases. Here we are going to rewrite general q as $\frac{2}{1-\epsilon}$ where $\epsilon \in [0, 1)$. The following power expansion might be helpful, before doing further calculation.

$$x^{2/q} = x^{1-\epsilon} \sim x - \epsilon x \log(x) + \frac{\epsilon^2}{2} x \log^2(x) \tag{3-12}$$

And x here means $\frac{|f'(\tau_1) f'(\tau_2)|^{\frac{1}{2}}}{|f(\tau_1) - f(\tau_2)|}$, the log part can also be expanded as

$$-\log|\tau_1 - \tau_2| - \frac{1}{8} \frac{|f''(\tau_2)|^2}{|f'(\tau_2)|^2} |\tau_1 - \tau_2|^2 + \frac{1}{12} \frac{|f'''(\tau_2)|}{|f'(\tau_2)|} |\tau_1 - \tau_2|^2 + \cdots \tag{3-13}$$

Meanwhile, the $b(q)$ can be expanded as follows (This is the $b(q)$ from SYK_q we'll discuss our result later)

$$\frac{1}{\pi J} + \frac{\epsilon \log(\pi J)}{\pi J} + \frac{\epsilon^2 \left(12 \log^2(J) + 24 \log(\pi) \log(J) - \pi^2 + 12 \log^2(\pi)\right)}{24 \pi J} \tag{3-14}$$

Putting these two parts together, we'll have

$$\partial_{\tau_1} \cdot \left(\frac{1}{\tau_{12}} + \frac{A\tau_{12}}{12} - \frac{B\tau_{12}}{8} - \epsilon \left(\frac{A\tau_{12}}{12} - \frac{B\tau_{12}}{8} + \frac{1}{\tau_{12}} \right) \left(\frac{A\tau_{12}^2}{12} - \frac{B\tau_{12}^2}{8} - \log(\tau_{12}) \right) + \cdots \right)$$
$$= -\frac{1}{\tau_{12}^2} + \frac{A}{12} - \frac{B}{8} + O_{AB}(\epsilon, \tau_{12}) + O(\epsilon, \tau_{12})$$

With $A = \frac{|f'''(\tau_2)|}{|f'(\tau_2)|}$, $B = \frac{|f''(\tau_2)|^2}{|f'(\tau_2)|^2}$ What we do here is the double expansion over τ_{12} and regularization parameter ϵ, and O_{AB} represent higher term in t, ϵ coupled with dynamic mode $f(\tau)$. We can ignore them since they are next order correction.O means the one that did not coupled with $f(\tau)$. We can throw them, as long as the one $\frac{1}{\tau_{12}}$, they did not describe the dynamic of our system can proper renormalization would cancel these apart. And we can see this procedure indeed gives us the Schwarzian we want where we use the definition on Schwarzian as follows.

$$\boxed{\text{Sch}(\tau(u), u) \equiv \frac{2\tau'\tau''' - 3\tau''^2}{2\tau'^2}} \tag{3-15}$$

Our derivation above only arrives at the Schwarzian form in a simple manner, but for the specific coefficients of the action, some numerical calculations are required, which we denote as $\alpha_S/\tilde{J}$, and $\{f(u), u\}$ means $\text{Sch}(f(u), u)$. Here, we follow the notation used by Maldacena[4].

$$\boxed{S = -\frac{N\alpha_S}{\mathcal{J}} \int du \{f(u), u\}} \tag{3-16}$$

With $\mathcal{J}$ follow $\frac{J^2(q-1)!}{N^{q-1}} = \frac{2^{q-1}}{q} \frac{\mathcal{J}^2(q-1)!}{N^{q-1}}$

3.2 JT Gravity

3.2.1 Deriving Schwarzian Action in JT gravity

Since JT gravity is a broader field in theoretical physics, and the author's main task is to explore the equivalence between SYK and JT at the level of action, the form of the JT gravity action is directly given here. For more discussions on JT gravity, one can refer to the following papers[21][14].

$$I = -\frac{\phi}{16\pi G}\left[\int_M d^2x\sqrt{g}R + 2\int_{\partial M} K\right] \tag{3-17}$$

$$-\frac{1}{16\pi G}\left[\int_M d^2x\sqrt{g}\phi(R+2) + 2\int_{\partial M}\phi_b K\right] \tag{3-18}$$

In which the first part is the Gauss-Bonnet term, defined by Gauss-Bonnet, and the integral over the manifold M is determined by the Euler characteristic χ of the manifold M. It can be seen that this is a topological property of the manifold M, and there is no dynamics we are looking for.

The second line allows us to obtain the dynamics at the boundary points of AdS_2.After integration over ϕ_{bulk} Or say using the EOM given by ϕ, we are dealing with boundary Action in $R = -2$ which is AdS_2 background spacetime.

$$\begin{aligned} I_{bdy} &= -\frac{1}{8\pi G}\int_{\partial M}\phi K \\ &= -\frac{1}{8\pi G}\int du\sqrt{g_{uu}}\frac{\phi_r}{\epsilon}K \\ &= -\frac{1}{8\pi G}\int du\frac{\phi_r}{\epsilon^2}K \end{aligned}$$

ϵ is a UV cutoff parameter who is very very tiny, demonstrating how close we get to the boundary of AdS_2. Detailed discussion of it can be followed in the reference mentioned above. Here, the second line is owing to the fact that dilaton field ϕ diverge as $\frac{\phi_r}{\epsilon}$ at boundary; and the last line is achieved by the boundary condition that $g_{\text{bdy}} = \frac{1}{\epsilon^2}$. Before Calculating K, we need to define T^μ and n^ν as tangent vector and normal vector for convenience.

Before further calculating extrinsic curvature K, we have to make following definination at first.

Tangent vector's,Extrinsic Curvature's and Normal vector's definition:

$$K \equiv g^{\mu\nu}\nabla_\mu n_\nu T^\mu \equiv (t'(u), z'(u))T^\mu n_\mu \equiv 0, n^\mu n_\mu \equiv 1 \tag{3-19}$$

Its easily solved that $n^\mu = \frac{z}{\sqrt{t'^2+z'^2}}(-z', t')$

Rewrite $g^{\mu\nu}$ as follows,

$$K \equiv g^{\mu\nu}\nabla_\mu n_\nu = (\frac{T^\mu T^\nu}{T^2} + n^\mu n^\nu)\nabla_\mu n_\nu \tag{3-20}$$

Since $\nabla_\mu |n|^2 = 2n^\nu \nabla_\mu n_\nu, \nabla_\mu |n|^2 = 0$, we have

$$\begin{aligned} K &= \frac{T^\mu T^\nu}{T^2}\nabla_\mu n_\nu \\ &= \frac{T^\nu}{T^2}\left(T^\mu \partial_\mu n_\nu - \Gamma^\rho_{\mu\nu} n_\rho T^\mu\right) \end{aligned}$$

Since $T^\mu = (\partial_u t, \partial_u z)$ and we can write $T^\mu \partial_\mu$ as ∂_u which would be helpful in future to simplify our expression.

1. Calculate $\frac{T^\nu}{T^2}\partial_u n_\nu$ part

Under Poincare Coordinate, our metric is as follows, which can be used to lower the index of $n^\mu = \frac{z}{\sqrt{t'^2+z'^2}}(-z', t')$

$$ds^2 = \frac{dt^2 + dz^2}{z^2} \tag{3-21}$$

And we'll have $n_\nu = \frac{1}{z\sqrt{t'^2+z'^2}}(-z', t')$

Since $T^\mu = (t', z')$, Then we have

$$T^2 = g_{\mu\nu}T^\mu T^\nu = \frac{1}{z^2}(t'^2 + z'^2) \tag{3-22}$$

Together we have

$$T^\nu \partial_u n_\nu = \frac{z't'' - t'z''}{z\sqrt{t'^2 + z'^2}} \tag{3-23}$$

And we'll arrive

$$\frac{T^\nu}{T^2}\partial_u n_\nu = z\frac{z't''-t'z''}{(t'^2+z'^2)^{3/2}} \tag{3-24}$$

2. Christoffels Part

Now dealing with Christoffels, it can be computed from the metric that

$$-\Gamma^t_{tz} = -\Gamma^t_{zt} = \Gamma^z_{tt} = -\Gamma^z_{zz} = \frac{1}{z} \tag{3-25}$$

Putting them back in the K formula

$$\begin{aligned}
\Gamma^\rho_{\mu\nu}n_\rho T^\mu T^\nu &= 2\Gamma^t_{tz}n_t T^t T^z + \Gamma^z_{tt}n_z T^t T^t + \Gamma^z_{zz}n_z T^z T^z \\
&= -\frac{2}{z}\frac{-z'}{z\sqrt{t'^2+z'^2}}t'z' + \frac{1}{z}\frac{t'}{z\sqrt{t'^2+z'^2}}t'^2 - \frac{1}{z}\frac{t'}{z\sqrt{t'^2+z'^2}}z'^2 \\
&= \frac{1}{z^3\sqrt{t'^2+z'^2}}(2z'^2t' + t'^3 - z'^2t') \\
&= \frac{1}{z^3\sqrt{t'^2+z'^2}}(z'^2t' + t'^3)
\end{aligned}$$

Combine those parts together with correct sign, we'll arrive

$$K = \frac{t'(t'^2+z'^2+zz'') - zz't''}{(t'^2+z'^2)^{3/2}} \tag{3-26}$$

Under limit of $\epsilon \to 0$ we'll claim that $z \cong \epsilon t'$. Putting it back and save the leading term ϵ^2 we have

$$K = 1 + \epsilon^2\frac{2t't''' - 3t''^2}{2t'^2} \equiv 1 + \epsilon^2\mathrm{Sch}(t(u),u) + O(\epsilon^4) \tag{3-27}$$

Neglecting the field independent divergent term, we'll see the near AdS_2 bounday is governed by the dynamic of Schwarzian action. And the effective action for JT gravity near it boundary has the form of

$$I_{JT} = -\frac{1}{8\pi G_N}\int du\phi_r(u)\mathrm{Sch}(t(u),u) \tag{3-28}$$

with notation following[14] where S means Sch and $\phi_r(u)$ is a source for the operator dual to the dilaton. We can see that we have the same effective action as SYK in above.

3.3 Important Mathematic Properties For Schwarzian

Before solving the EOM of Schwarzian to discover the dynamic of the system, we'll consider the mathematical propertiesd of Schwarzian at first.

3.3.0.1 Comfosition law

It can be verified directly use mathematica in Appendix C that when we compose the reparametrization mode $f \to f \circ g$,

$$\mathrm{Sch}(f \circ g, t) = g'^2 \mathrm{Sch}(f, g) + \mathrm{Sch}(g, t). \tag{3-29}$$

This property would be helpful in deriving following finite temperature transformation.

3.3.0.2 Finite Temperature Transformation:

Since we are discussing in Euclidean time, and the period of time can be considered as the inverse of temperature. Given transformation between $\tau(u)$ and $t(u)$ as follows: $t(u) = \tan\frac{\tau(u)}{2}$ where $\tau \in [-\frac{\pi}{2}, \frac{\pi}{2})$. It can be considered as a finite temperature transformation since it maps infinite Euclidean time t to finite time τ. It can also be seen as changing the Euclidean time to Rindler time, where the periodicity is involved to get rid of the conical singularity[22].This transformation also builds the bridge between Poincare coordinate and Rindler coordinate.

We can represent $t(u)$'s Schwarzian by using Schwarzian of $\tau(u)$, which can be easily checked.

$$\mathrm{Sch}(t(u), u) = \mathrm{Sch}(\tau(u), u) + \frac{1}{2}\tau'^2 \tag{3-30}$$

We can therefore write our action in terms of Rindler Coordinate, which helps in discovering BH physics.

$$I_{bdy} = -C \int du \left[\mathrm{Sch}(\tau, u) + \frac{1}{2}\tau'^2\right] \tag{3-31}$$

C is a constant determined by the given scenario.

3.3.0.3 $SL(2, \mathbb{R})$

symmetry: We have seen in SYK case that Schwarzian action preserves the symmetry after the conformal symmetry breaks down. However, the conformal symmetry does not breaks in to ashes, there are still $SL(2,\mathbb{R})$ symmetry alive. We can easily check using Mathematica in Appendix C that when we are doing $\tilde{t}(u) = \frac{at(u)+b}{ct(u)+d}$ with $ad - bc \equiv 1$ would preserve the form of Schwarzian. The $SL(2,\mathbb{R})$ condition would restrict the integration measure unchanged.

3.4 EOM of Schwarzian

We have already seen from above that in near boundary region for JT gravity and in the region away from conformal sector, both of them has the same effective description as the Schwarzian action. Actually, we'll soon going to see that its not a coincidence. Schwarzian action serve as the simplest form that preserve $SL(2,\mathbb{R})$ symmetry. There are also a lot of discussion over Schwarzian and the redundancy it carries on needs to be gauged according to the discussion in[21]. However, we don't need to worry about those at present. In this section, we'll discuss the dynamics and mathematical properties of Schwarzian action. Following discussion can be mainly found in[21][14].

3.4.1 Equation of motion Schwarzian action

EOM (equation of motion) can be easily calculated in most cases, using Euler-Lagrange eqn. However, it needs the assumption that $\mathcal{L}$ contains field derivative up two second time. However, we can see Sch has derivative up to third time. Therefore we need to dervive the EOM from variation principle. We'll do variation over $t(u)$

$$
\begin{aligned}
\int \phi \mathrm{Sch}(t(u),u) &= \int \phi\left(\frac{t''}{t'}-\frac{3}{2}\left(\frac{t''}{t'}\right)^2\right)\\
&= \int \phi'\left(\frac{t''}{t'}\right)' - \phi\frac{1}{2}\left(\frac{t''}{t'}\right)^2\\
&= -\int \phi'\frac{t''}{t'}+\phi\frac{1}{2}\left(\frac{t''}{t'}\right)^2\\
&= \int -\left[\phi'\cdot\frac{\delta t''}{t'}-\phi'\frac{t''}{t'^2}\delta t'+\phi\frac{t''\delta t''}{t'^2}-\phi\frac{t''^2}{t'^3}\delta t'\right]\\
&= \int -\left[\delta t''\cdot\left(\frac{\phi'}{t'}+\phi\frac{t''}{t'^2}\right)-\delta t'\cdot\left(\phi'\frac{t''}{t'^2}+\phi\frac{t''^2}{t'^3}\right)\right]\\
&= \int -\left[\left(\frac{\phi'}{t'}+\phi\frac{t''}{(t')^2}\right)''+\left(\phi'\frac{t''}{t'^2}+\phi\frac{t''^2}{t'^3}\right)'\right]\delta t\\
&= -\int \left[\left(\frac{\phi' t'}{(t')^2}\right)''+\left(\frac{t''}{t'^3}(\phi t')\right)'\right]\delta t\\
&= -\int \left[\left(\frac{(\phi t')'}{t'^2}\right)'+\left(\frac{t''}{t'^3}(\phi t')'\right)\right]'\delta t
\end{aligned}
$$

Dealing with the integrand and strip the total derivative. Denote $(\phi t')' \equiv X$ for convenience.

$$
\begin{aligned}
\left(\frac{X}{t'^2}\right)'+\frac{t''}{t'^3}X &= \frac{t'^2X'-2t't''X}{t'^4}+\frac{t''}{t'^3}X\\
&= \frac{1}{t'^2}X'-\frac{t''}{t'^3}X\\
&= \frac{1}{t'}\left(\frac{X}{t'}\right)'\\
&= \frac{1}{t'}\left(\frac{(t'\phi)'}{t'}\right)'
\end{aligned}
$$

Therefore the equation of motion for Schwarzian action is

$$
\left[\frac{1}{t'}\left(\frac{(t'\phi_r)'}{t'}\right)'\right]' = 0 \tag{3-32}
$$

3.4.2 Different Form Of Schwarzian EOM

In $I = \int \phi \mathrm{Sch}(t(u),u)$ discussed above, constant ϕ is the type of action we'll mainly discuss about. We would derive reparametrization mode as the dynamic of the system by asking

$$\frac{\delta I}{\delta t(u)} = 0 \tag{3-33}$$

Therefore we are looking for

$$\frac{\delta \mathrm{Sch}}{\delta t} = \frac{\frac{\delta \mathrm{Sch}}{\delta u}}{\frac{\delta t}{\delta u}} = \frac{\mathrm{Sch}'}{t'} = 0 \tag{3-34}$$

And that means we are looking for $\mathrm{Sch}(t(u), u) = \mathrm{const.}$

Using composition law can easily verify that if $\tau(u)$ is linear in u, Schwarzian is constant. This result satisfies as a special solution for Schwarzian Action.

Since u is time on the boundary and it would also be periodic if τ is linear in u. Since τ is Rindler time and in periodic of 2π. We would set the time periodic of boundary as β and a simple relation between τ and u satisfying EOM of Schwarzian would be

$$\tau(u) = \frac{2\pi}{\beta} u \tag{3-35}$$

and Schwarzian would be

$$I_{\mathrm{Sch}} = -2\pi^2 C \frac{1}{\beta}, \quad C = \frac{\bar{\phi}_r}{8\pi G_N}. \tag{3-36}$$

Chapter 4 JT Gravity, SYK, and MQ model

We have found that the JT/SYK model has a duality in the infrared region. When we consider the JT gravity in AdS_2 space, there are two boundaries, which should correspond to two decoupled SYK systems. Gravity side, introducing a coupling in the JT gravity can violate the Average Null Energy Condition (ANEC) and form a traversable wormhole. The corresponding SYK systems, on the field theory side, should also introduce two-sided coupling. This chapter describes the corresponding IR dynamics, and we will find that they exhibit consistent behavior when using different units for u. Our discussion in this chapter focuses on building the correspondence between the gravity side and the field theory side. A detailed discussion on its dynamics is saved to the next chapter.

4.1 Gravity side

Before delving into a detailed discussion of JT gravity on two boundaries, we must first review the geometry AdS_2. Subsequently, we will calculate the mathematical process of introducing coupling to achieve a traversable wormhole and obtain the IR action on the gravity side. As for the violation of ANEC due to the introduction of coupling, we will not elaborate much on this here; for more details, refer to[23].

4.1.1 2D Space, Time and Coordinate

When we previously derived JT gravity, we used Poincaré time and boundary time in description, which belong to different coordinate systems. We only roughly described that there can be a certain mapping relationship between the two, but we did not specify the exact impact of the reparameterization of boundary time on Poincaré time. Moreover, the AdS_2 spacetime has more than one boundary. Therefore, before further discussing the dynamics, it is necessary to carefully discuss the coordinate system of AdS_2.

Coordinates on AdS_2

The Penrose diagram of AdS_2 has two boundaries. To obtain the transformations between different coordinate systems of this spacetime, we need to introduce the discussion of embedding coordinates to derive the relationship between global and Poincaré coordinates.

Relevant content is referenced from[24].

AdS_2 , a curved spacetime, can be regarded as a hypersurface embedded in a three-dimensional flat spacetime with the metric $(-1, -1, 1)$:

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 = -\frac{1}{\mu^2},$$
$$ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2.$$

It satisfies $R = -2\mu^2$. Since the action in JT gravity shows that we are discussing a spacetime with $R = -2$, we choose $\mu = 1$ here, and we will substitute $\mu = 1$ into the discussion at the end.

Global Coordinate

When we use the following transformations, we obtain the Global Spacetime Coordinates with $x \in [\sigma, \pi/\mu]$, $t \in \mathbb{R}$:

$$\begin{aligned}
Y_{-1} &= \frac{1}{\mu}\frac{\cos(\mu t)}{\sin(\mu\sigma)},\\
Y_0 &= \frac{1}{\mu}\frac{\sin(\mu t)}{\sin(\mu\sigma)},\\
Y_1 &= \frac{1}{\mu}\cot(\mu\sigma),\\
ds^2 &= \frac{1}{\sin^2(\mu\sigma)}(-dt^2 + d\sigma^2).
\end{aligned}$$

Poincaré Coordinate

When we use the following transformations, we obtain the Poincaré Coordinates, with $x \in (-\infty, 0)$, $t \in \mathbb{R}$:

$$\begin{aligned}
Y_{-1} &= -\frac{1}{2\mu}\left(\frac{1}{\mu x} + \mu x\right) + \frac{t^2}{2x},\\
Y_0 &= -\frac{t}{\mu x},\\
Y_1 &= -\frac{1}{2\mu}\left(\frac{1}{\mu x} - \mu x\right) - \frac{t^2}{2x},\\
ds^2 &= \frac{1}{\mu^2 x^2}(-dt^2 + dx^2).
\end{aligned}$$

Rindler Coordinate

When we use the following transformations, we obtain the Rindler coordinates:

$$
\begin{aligned}
Y_{-1} &= \frac{1}{\mu}\cosh(\mu\rho), \\
Y_0 &= \frac{1}{\mu}\sinh(\mu\rho)\sinh(\mu t_R), \\
Y_1 &= \frac{1}{\mu}\sinh(\mu\rho)\cosh(\mu t_R), \\
ds^2 &= -\sinh^2(\mu\rho)\, dt_R^2 + d\rho^2.
\end{aligned}
$$

We can easily see that when discussing boundary behaviors, we encounter divergences in the above coordinate systems. Therefore, we need to use the projective coordinates X^M to describe them instead. X^M is a projection modification of the above Y^M coordinates. We require $g_{MN}X^M X^N = 0$ and $X^M \sim \lambda X^M$.

Global Coordinate @ Boundary

$$
\begin{aligned}
X_{-1} &\sim \frac{1}{\mu}\cos(\mu t), \\
X_0 &\sim \frac{1}{\mu}\sin(\mu t), \\
X_1 &\sim \frac{1}{\mu}\cos(\mu\sigma).
\end{aligned}
$$

It can be verified that this satisfies the requirements of the projective coordinates. Substituting $\mu = 1$ and considering $\sigma = 0, \pi$ to obtain t_l and t_r, we obtain the relationship between the embedding coordinates and the global coordinates.

$$
\begin{aligned}
e^{it_r} &= X_{-1} + iX_0, \quad \text{for} \quad X_1 = 1, \\
e^{it_l} &= X_{-1} + iX_0, \quad \text{for} \quad X_1 = -1.
\end{aligned}
$$

Poincaré Coordinate @ Boundary

$$
\begin{aligned}
X_{-1} &= -\frac{1}{2\mu}\left(\frac{1}{\mu} + \mu x^2\right) + \frac{t^2}{2}, \\
X_0 &= -\frac{t}{\mu}, \\
X_1 &= -\frac{1}{2\mu}\left(\frac{1}{\mu x^2} - \mu x\right) - \frac{t^2}{2},
\end{aligned}
$$

Thus, we obtain the relationship between the embedding coordinates and the Poincaré coordinates:

$$\frac{X_0}{X_{-1} + X_1} = t_p \tag{4-1}$$

Rindler Coordinate @ Boundary

$$
\begin{aligned}
X_{-1} &= \frac{1}{\mu}\coth(\mu\rho), \\
X_0 &= \frac{1}{\mu}\sinh(\mu t_R), \\
X_1 &= \frac{1}{\mu}\cosh(\mu t_R),
\end{aligned}
$$

It satisfies $X \cdot X = 0$. Also, one should notice when reaching $\rho \to \infty$, $X_{-1} = \lim_{\rho\to\infty}\coth(\rho) = 1$. Thus, we obtain the relationship between the embedding coordinates and the Rindler coordinates as follows:

$$e^{t_R} = X^1 + X^0, \quad \text{for} \quad X^{-1} = 1 \tag{4-2}$$

Therefore, the transformation relations between the global time, Rindler time, and Poincaré time of the AdS_2 space are as follows:

$$\boxed{\frac{X^0}{X^{-1} + X^1} = t_P = \tan\frac{t_r}{2} = -\frac{1}{\tan\frac{t_l}{2}} = \tanh\frac{t_R}{2}} \tag{4-3}$$

Proof:

When $X^1 = \pm 1$, we have

$$\cos(t_{l,r}) = X^{-1} \quad \text{and} \quad \sin(t_{l,r}) = X^0. \tag{4-4}$$

Substituting them into the embedding coordinate expression of t_P, we obtain

$$t_P = \frac{X^0}{X^{-1} + X^1} = \frac{\cos(t_{l,r})}{\sin(t_{l,r}) \pm 1} = \tan\left(\frac{t_r}{2}\right) = -\frac{1}{\tan\left(\frac{t_l}{2}\right)}. \tag{4-5}$$

When we are looking at Rinder coordinate, we have the following identity:

$$-1^2 - (X^0)^2 + (X^1)^2 = 0 \Rightarrow (X^1 + X^0)(X^1 - X^0) = 1. \tag{4-6}$$

By substituting X^1 and X^0 of Rindler coordinate into above equation, we get

$$e^{t_R} = X^1 + X^0 \quad \text{therefore} \quad e^{-t_R} = \frac{1}{X^1 + X^0} = X^1 - X^0. \tag{4-7}$$

Solving these equations yields

$$X^1 = \cosh(t_R) \quad \text{and} \quad X^0 = \sinh(t_R). \tag{4-8}$$

Finally, substituting them in expression for t_P we arrive at

$$t_P = \tanh\left(\frac{t_R}{2}\right) \qquad QED. \tag{4-9}$$

Remark: Those relationships are of great significance for the subsequent discussion of Physical Dynamics!.

Remember that when we derived the JT gravity with the raising and lowering of indices, we used the metric corresponding to the Poincaré metric. We denote the time in JT gravity as t_P, and u is the boundary time. Physical meaning of the reparameterization mode $t_P(u)$ is a mapping between the boundary time and the interior Poincaré time. And the effective action is given by

$$S = -\phi_r \int \{t_P(u), u\} du \tag{4-10}$$

In Chapter 3, we discussed in Euclidean signature, one solution obtained is $\tau = \frac{2\pi}{\beta}u$. The dynamics we are discussing now uses the Lorentzian signature, so we need to perform a Wick rotation to obtain the desired result. Based on the previously discussed relationship between t_R and t_P, we can obtain

$$t_P = \tanh\frac{t_R(u)}{2} = \tanh\frac{\pi u}{\beta} \tag{4-11}$$

4.1.2 Dynamics of JT gravity in AdS_2

When spacetime has two boundaries, we have two sets of mappings. According to the analysis of JT gravity, dynamics only exist on the boundaries. Therefore, when we describe the dynamics of the system, we should have two copies of Schwarzian, which is Sch_L+Sch_R[6]. We aim to construct a traversable wormhole, so we need to introduce a coupling term on the two boundaries[23],[6]. Whether this construction can give rise to an eternal traversable wormhole is a relatively more complex topic, which only slightly mentioned in the next chapter. The setup here is to introduce the coupling term[6][23]

$$S_{\text{int}} = g\sum_{i=1}^{N}\int du\, O_L^i(u)O_R^i(u) \tag{4-12}$$

where $O(u)$ is a bulk operator of dimension Δ, evaluated at the boundary. We choose the coupling g and $\frac{1}{N}$ as small quantities, but keep Ng fixed, similar to the general AdS/CFT settings. Since g is a small quantity, we can use the following approximation:

$$\left\langle e^{ig\sum_i\int du\, O_L^i(u)O_R^i(u)}\right\rangle \sim e^{ig\sum_i\int dt\langle O_L^i(u)O_R^i(u)\rangle} \tag{4-13}$$

This approximation arises from the fact that in large N theories, the ladder diagram is the leading contribution, but we will not discuss these Feynman diagrams in detail here. We choose the normalized form of the two-point function as follows. If we reparameterize the two-point function with $t \Rightarrow \tilde{t}$, we obtain the following relation:

$$\langle O(t_P^1)O(t_P^2)\rangle = |t_P^1 - t_P^2|^{-2\Delta} \Rightarrow \frac{(\tilde{t}_1'\tilde{t}_2')^{\Delta}}{|\tilde{t}_1 - \tilde{t}_2|^{2\Delta}} \tag{4-14}$$

Using the previously discussed relationship between t_P and $t_{l,r}$, we can express the single interaction term in the large N limit using the boundary dynamical modes. Specifically, by substituting $t_P = \tan\left(\frac{t_r}{2}\right) = -\frac{1}{\tan\left(\frac{t_l}{2}\right)}$, we obtain

$$\langle O(t_P^1)O(t_P^2)\rangle = \frac{1}{2^{2\Delta}}\left(\frac{t_l'(u)t_r'(u)}{\cos^2\frac{t_l(u)-t_r(u)}{2}}\right)^{\Delta} \tag{4-15}$$

Including the factor of $g\sum\limits_i$ and dynamics on each boundary, we get the complete action on the JT gravity side as follows:

$$S = \int du\left[-\phi_r\left\{\tan\frac{t_l(u)}{2}, u\right\} - \phi_r\left\{\tan\frac{t_r(u)}{2}, u\right\} + \frac{gN}{2^{2\Delta}}\left(\frac{t_l'(u)t_r'(u)}{\cos^2\frac{t_l(u)-t_r(u)}{2}}\right)^{\Delta}\right] \tag{4-16}$$

4.2 Field Theory Side

We are going to obtain a similar action to eqn 4-16 , which is pretty easy by replacing the correct reparametrization mode. However, we are going to set up the discussion on the SYK side from its basement. Here, we will first discuss why we often refer to TFD later on, and how its construction is very similar to that of the Keldysh SYK. Based on TFD, we construct the $\text{SYK}_{L,R}$ that we need, and then discuss the IR dynamics after introducing the coupling term, ultimately finding that it formally behaves consistently with the gravity side.

4.2.1 The Significance of TFD in Holographic Duality

To understand why we need to discuss TFD, we must recognize that JT gravity, as an AdS_2 gravity, naturally possesses a two-sided black hole description. This is discussed in[25] and[21] regarding the relationship between JT gravity and the two-sided black hole. In[26], it is discussed that the field theory description corresponding to the Eternal Black hole (which has the same spacetime structure as the two-sided black hole in AdS_2) is the TFD state. Therefore, when discussing the coupled SYK and its holographic duality properties, we should start from the TFD state. •

As for the discussion of the two-sided black hole in JT gravity, it will not be reiterated here; the connection between the Eternal Black hole and the TFD state can be referred to in[22], which provides some intuitive descriptions of the association between the eternal black hole and TFD.

4.2.2 The Construction of TFD

The general construction of TFD is carried out in the following manner. Consider two identical systems denoted as H_1 and H_2, with β representing the time period and $Z(\beta)$ being the partition function of the system. The TFD state can be written as

$$|TFD\rangle \equiv \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_1 |n\rangle_2 \tag{4-17}$$

We assume the total Hamiltonian of the system to be $H_{tot} = H_1 - H_2$, and it can be observed that the TFD state does not evolve with time.

$$|TFD(t)\rangle = \sum_n e^{-\beta E_n/2} e^{-i(H_1 - H_2)} |n\rangle_1 |n\rangle_2 = |TFD\rangle \tag{4-18}$$

In the SYK case we are considering, we define TFD state in a similar way.

$$|TFD_\beta\rangle \equiv Z_\beta^{-1/2} e^{-\beta(H_L+H_R)}|I\rangle \tag{4-19}$$

When we take $\beta \to 0$, we obtain a natural interpretation of $|I\rangle$ as the infinite temperature TFD state. Also, the chosen total Hamiltonian of the system is $H_L - H_R$. To ensure that $|I\rangle$ is time translationally invariant, we must require:

$$(H_L - H_R)|I\rangle = 0 \tag{4-20}$$

We still have many choices for $|I\rangle$. If we demand that $|I\rangle$ is the ground state annihilated by $f^i \equiv \frac{\psi_L^i + i\psi_R^i}{\sqrt{2}}$, then we have following result. (Similar result would also be derived in Choi-Jamiolkowski isomorphism, seen in chapter 7 for detailed discussion) Since the combination of majorana fermions gives dirac fermioins, $|I\rangle$ can be seen as a ground stated defined by annihilating of dirac fermions.

$$\left(\psi_L^j + i\psi_R^j\right)|I\rangle = 0 \tag{4-21}$$

Although we set H as the Hamiltonian of a copy of the SYK system, which has the following form, H is not fully fixed because the sign of its coupling has a $\mathbb{Z}_2$ redundancy.

$$H = (i)^{q/2} \sum_{1\le j_1 \le j_2 \cdots \le j_q} J_{j_1 j_2 \cdots j_q} \psi^{j_1}\psi^{j_2}\cdots\psi^{j_q} \langle J^2_{j_1\cdots j_q}\rangle = \frac{2^{q-1}\mathcal{J}^2(q-1)!}{qN^{q-1}} \quad , \quad \mathcal{J} = \frac{\sqrt{q}J}{2^{\frac{q-1}{2}}} \tag{4-22}$$

We can see that if we set $J^R_{\dots} = (-1)^{\frac{q}{2}} J^L_{\dots}$, then we'll have $H_R = (-1)^{\frac{q}{2}} H_L$. And we can see it is a ground state of decoupled SYK system. Disregarding those exponential factors which can be commuted by $H_L - H_R$, we can act $H_L - H_R$ on $|I\rangle$). And we see that the time translational invariance of $|I\rangle, |TFD\rangle$ is satisfied.

$$\begin{aligned} H_L - H_R|I\rangle &= H_L - \sum J^R_{j_1\cdots j_q}\psi_R^{j_1}\cdots\psi_R^{j_q}|I\rangle \\ &= (1 - (-1)^{\frac{q}{2}} \cdot (-i)^q) \times H_L|I\rangle \\ &= 0 \end{aligned} \tag{4-23}$$

4.2.3 The Construction of H_{total}

Similar to the discussion in JT gravity, we introduce an interaction term with a small coupling coefficient μ in the doubled SYK, and ultimately obtain an effective action consistent with JT gravity in different units.

$$H_{\text{total}} = H_{\text{L,SYK}} + H_{\text{R,SYK}} + H_{\text{int}} \ , \ H_{\text{int}} = i\mu \sum_j \psi_L^j \psi_R^j \tag{4-24}$$

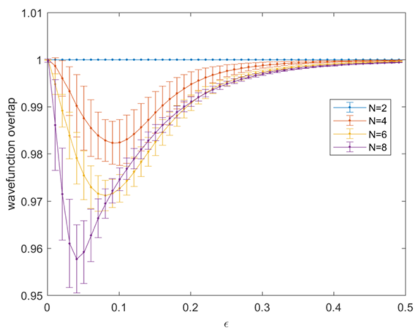

Figure 4-1 The overlap $|\langle TFD|G\rangle|$ for $N = 4, 8, 12, 16$ Majorana fermions per site.

One should be aware that $\psi^2 = \frac{1}{2}$ here instead of 0 shwon in chapter 2. The reason is that we have already introduced the quantization condition $\{\psi_a^i(t), \psi_b^j(t)\} = \delta_{ab}\delta^{ij}$ for discussion over energy. Related discussion on quantization condition can also be seen in Appendix B.

It is claimed that $|I\rangle$ is the ground state of H_{int} and H_{tot}. And we have

$$i\mu \sum \psi(\tau)_L^i \psi(\tau)_R^i |I\rangle = -\mu \sum \psi(\tau)_L^i \psi(\tau)_L^i |I\rangle = -\frac{\mu N}{2}$$

.

However, $[H_{int}, e^{-\beta(H_L+H_R)}] \neq 0$. Luckily, we are discussing interaction with $\mu \ll 1$, and we can think that the ground state $|G\rangle$ of H_{total} is approximately $|TFD_\beta\rangle$, and the value of $\beta = \beta(\mu)$ can be solved by minimizing the energy. As for how close it would be, would not be discussed in this paper. We only show the result in [6].

4.2.3.1 Discussion on TFD state and ground state

Giving N and q, the Hamiltonian can be constructed using iteration method mentioned in[14] and appendix in[27]. Therefore we would have a matrix representation of the total Hamiltonian. Through diagonalization we can obtain the energy and corresponding eigenstate $|n\rangle$ of single SYK system. Therefore, we can construct $|TFD\rangle$ defined in eqn 4-17 as follows.

On the other hand, manipulation over Hamiltonian in matrix representation can give us the description $|G\rangle$. Therefore those two states are comparable numerically.

As for why bother using TFD state. Since it is as convenient as vaccum state, nor ground state. One of the reason is briefly mentioned in the beginning of this section. Now we are going to elaborate this point a little bit clearly. In[26], it is mentioned TFD is dual to eternal black hole, or eternal black hole shown in figure 4-2.

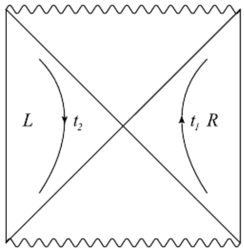

Figure 4-2 Penrose diagram of eternal black hole[23]

A naive way for understanding : We draw two black holes on a single diagram to describe the entanglement between the two black holes. The perturbation of the spacetime geometry allows us to send a signal from one black hole to the other in a decoupled system. However, when we do not perform a double trace deformation, we cannot obtain any information by observing just one of the black holes. The corresponding field theory description is that taking the partial trace of $|TFD\rangle$ results in a maximally entangled state with no information decoded.

4.2.4 Low Energy Region and Schwarzian

For this system, there are several interesting small parameters in the region under consideration. Specifically, in the decoupled system, it is believed that conformal symmetry is exhibited when energy $\ll J$. In the coupled system, another small coupling μ is introduced. Both of them are small, making it tricky in perturbation theory. Here, we believe that the expansion of these two small quantities can be handled in the following way: using perturbation theory to describe H_{int} in conformal region. The specific low-energy description is as follows:

- H_{SYK} is described by the Schwarzian Action.
- H_{int} is approximated to $\langle H_{int}\rangle$ in the same way as the JT gravity part

Following above agreements, we can use the conformal approximation to obtain $\langle \psi_L(t_P^L)\psi_R(t_P^R)\rangle$ shown in H_{int}. Though we expanded $G(t_1, t_2)$ in terms of t_{12} and kept the leading term in our discussion of the Schwarzian, It is suppressed by the g factor in H_{int}. We'll just leave it there with no expansioin on t_12. Moreover, we adopt the same time reparametriization mode as gravity side: $t_P = \tan\frac{t_r}{2}, \quad t_P = -\frac{1}{\tan\frac{t_l}{2}}$.

Since $\langle\psi(\tau_1)\psi(\tau_2)\rangle_{\text{conformal}} = c_\Delta \text{sgn}(\tau_{12})\frac{i}{\mathcal{J}|\tau_{12}|^{2\Delta}}$ is a conformal correlator in Euclidean space, we can use wick rotation and the above reparametrization mode to derive $\langle \sum_j \psi_L^j(t_l)\psi_R^j(t_r)\rangle$

$$\because \mathrm{sgn}(\tan\frac{a}{2}+\frac{1}{\tan\frac{b}{2}})=1$$
$$\left[\frac{1}{(\tan\frac{a}{2}+\frac{1}{\tan\frac{b}{2}})^2}\right]^{\Delta}\left[\frac{1}{\cos^2\frac{a}{2}\cdot\sin^2\frac{b}{2}}-\frac{1}{4}\right]^{\Delta}=\left[\frac{1}{4\cos^2\left(\frac{a-b}{2}\right)}\right]^{\Delta}$$
$$\therefore \langle\psi_L(t_P^L)\psi_R(t_P^R)\rangle = c_\Delta\frac{i}{[2\mathcal{J}\cos\frac{t_P^L-t_P^R}{2}]^{2\Delta}}$$

The final result is

$$S=N\int du\left\{-\frac{\alpha_S}{\mathcal{J}}\left(\left\{\tan\frac{t_l(u)}{2},u\right\}+\left\{\tan\frac{t_r(u)}{2},u\right\}\right)+\mu\frac{c_\Delta}{(2\mathcal{J})^{2\Delta}}\left[\frac{t_l'(u)t_r'(u)}{\cos^2\frac{t_l(u)-t_r(u)}{2}}\right]\right\} \tag{4-25}$$

Here we'll give the united form of dynamics both for JT and SYK

$$S=N\int d\tilde{u}\left\{-\left(\left\{\tan\frac{t_l(\tilde{u})}{2},\tilde{u}\right\}+\left\{\tan\frac{t_r(\tilde{u})}{2},\tilde{u}\right\}\right)+\eta\left[\frac{t_l'(\tilde{u})t_r'(\tilde{u})}{\cos^2\frac{t_l(\tilde{u})-t_r(\tilde{u})}{2}}\right]^{\Delta}\right\} \tag{4-26}$$

$$\tilde{u}\equiv\frac{\mathcal{J}}{\alpha_S}u=\frac{N}{\phi_r}u\,,\quad \eta\equiv\frac{\mu\alpha_S}{\mathcal{J}}\frac{c_\Delta}{(2\alpha_S)^{2\Delta}}=\frac{g}{2^{2\Delta}}\left(\frac{N}{\phi_r}\right)^{2\Delta-1} \tag{4-27}$$

How to solve the system would be partly discussed in next section.

Chapter 5 Continue Discussion on MQ Model

We have already discussed how the low-energy effective action is derived in the MQ model. Now, we need to discuss how to solve the low-energy effective action. We can find a linear solution through a simple argument. More comprehensive solutions need a discussion of gauge fixing. We will obtain a new set of effective actions describing the SYK solution through gauge fixing, and after discussing these, we will further determine the expression for $\beta(\mu)$ in $|TFD_\beta\rangle$, as well as the low E, low temperature limit solutions. Finally, we will verbally describe the different phases corresponding to the MQ model, without providing too many detailed explanations on it, since the main focus of my subject is to solve Schwarzian dynamics.

5.1 Linear Solution

Since $S/N = \mathrm{Sch}_{L,R} + \mathrm{int}$, and we have already discussed that the linear solution indeed is a solution for the Schwarzian action in previous chapter. Here, we verify that the linear solution is a solution for the whole action, under the gauge condition $t_l = t_r$ that will be discussed later.[21][6].

We'll perform a variation on the interaction part to prove the above statement.

$$
\begin{aligned}
\text{Variation:}\quad & \frac{\delta t_l' t_r' + t_l'\delta t_r'}{\cos^2} + t_l' t_r \cos^{-3}\cdot\sin\cdot(\delta t_l - \delta t_r) \\
&= -\left(\frac{t_r'}{\cos^2}\right)'\delta t_l - \left(\frac{t_l'}{\cos^2}\right)'\delta t_r + t_l' t_r' \frac{\sin}{\cos^3}(\delta t_l - \delta t_r) \\
&= -\frac{\cos^2 t_r'' + \cos\cdot\sin\cdot(t_l' - t_r')t_r'}{\cos^4}\delta t_l - (l \leftrightarrow r) + t_l' t_r' \frac{\sin}{\cos^3}(\delta t_l - \delta t_r) \\
&= -\left[\frac{t_r''}{\cos^2} + \frac{\sin}{\cos^3}(t_r' t_l' - t_r'^2 - t_l' t_r')\right]\delta t_l \\
&\quad - \left[\frac{t_l''}{\cos^2} + \frac{\sin}{\cos^3}(t_l' t_r' - t_l'^2 + t_l' t_r')\right]\delta t_r
\end{aligned}
$$

When we adopt the linear ansatz and ask $t_l(u) = t_r(u)$, we have $\sin\left(\frac{1}{2}(t_l - t_r)\right) = 0, t_{l,r}'' = 0$, and we can see that the variation is zero. This means that the linear ansatz indeed solves the interaction part.

5.2 Gauge Fixing

Here, we will not rigorously derive the origin of the conserved charge, but we will introduce some aspects of where the $SL(2)$ gauge symmetry in the Schwarzian comes from. For related discussions, refer to[21][6].

However, it is worth noting why the interaction still preserves the $SL(2)$ symmetry instead of breaking it. It might be troublesom in deriving how this cos form is invariant over $SL(2)$ charge directly, since common $SL(2)$ invariant form is following, and discussed in appendix C.

$$\left(\frac{t_1' t_2'}{\sin^2(t_1 - t_2)}\right) \quad , \quad \frac{t_1' t_2'}{(t_1 - t_2)^2} \tag{5-1}$$

The interaction term in MQ model is originated from $\left(\frac{t_1' t_2'}{(t_1 - t_2)^2}\right)^{\Delta}$ and we therefore believe it possess $SL(2)$ symmetry. Owing to the mapping of AdS_2 coordinate, the specific transformation needs to be modified, though we are not showing the specific form in this paper.

5.2.1 Gauge Freedom

In the study of gauge freedom in Schwarzian, we need to perform a diffeomorphism on $\tau(u)$ in $\mathrm{Sch}(\tan(\frac{\tau(u)}{2}, u)$. Since we have linear ansatz for τ we can ask $u \cong \tau(u)$ by using proper β. After that, we'll add perturbation $\epsilon(u)$ with $\epsilon \ll 1$ as diffeomorphism

$$\tau(u) = u + \varepsilon(u) \tag{5-2}$$

Putting them back in the Schwarzian, we would have

$$\begin{aligned} &\frac{1}{2} + \epsilon'(u) + \epsilon'''(u) \\ &\quad - \frac{3}{2}\epsilon''(u)^2 + \frac{1}{2}\epsilon'(u)^2 - \epsilon^{(3)}(u)\epsilon'(u) \\ &\quad - 2\epsilon'''(u)\epsilon'(u)^2 + 3\epsilon'(u)\epsilon''(u)^2 + \ldots \end{aligned}$$

When we study the dynamics of $\epsilon(u)$, we can throw away the contribution from $\frac{1}{2}$. Any total derivative term would be canceled since we are dealing with integration on a periodic boundary. Focusing on the leading order contribution, we have

$$\left(\frac{1}{2}\varepsilon'^2 - \frac{1}{2}\varepsilon''^2 - (\varepsilon''\varepsilon')'\right) \tag{5-3}$$

And the effective action is

$$I_{\text{Sch}} = \frac{C}{2}\int_0^{2\pi} du\left(\varepsilon''^2 - \varepsilon'^2\right) \tag{5-4}$$

$$\delta I_{\text{Sch}} = -C\delta\varepsilon'(\varepsilon''' + \varepsilon') \tag{5-5}$$

The solution can be obtained as $\varepsilon = (\alpha e^{iu} + \beta e^{-iu}) + \gamma$. That's the diffeomorphism discussion on single SYK. When we consider the coupled SYK, we only need to focus more on the interaction term to see what the diffeomorphism is for the whole action.

Since

$$\frac{1}{\cos^2(\varepsilon)} \sim 1 + x^2 \tag{5-6}$$

$$t_l' t_r' \longrightarrow t_l' t_r'(1 + \delta t_l' \cdot t_r' + \delta t_r' \cdot t_l' + O(\varepsilon^2)) \tag{5-7}$$

in order to keep the overall interaction variation at second-order perturbation and above, we need to require that in the case of $t_l = t_r$, $\delta t_l' \cdot t_r' + \delta t_r' \cdot t_l' = 0$. Therefore, we can obtain the following matching relationship. The matching relationship for the purple translation transformation has not been proved yet.

$$\delta t_l = \varepsilon^0 + \varepsilon^+ e^{it_l} + \varepsilon^- e^{-it_l}, \quad \delta t_r = \varepsilon^0 - \varepsilon^+ e^{it_r} - \varepsilon^- e^{-it_r} \tag{5-8}$$

5.2.2 Conserved Charge

To satisfy the gauge fixing condition, we need to satisfy all $Q = 0$ conditions. The derivation of Q will not be discussed in more detail here; the following result comes from[6].

$$\begin{aligned}
Q_0/N &= Q_0^S[t_l] + Q_0^S[t_r] + \left(\frac{1}{t_l'} + \frac{1}{t_r'}\right)F, \\
F &\equiv \Delta\eta\left[\frac{t_l' t_r'}{\cos^2\frac{t_l - t_r}{2}}\right]^{\Delta}, \\
Q_0^S[t] &= -t' + \frac{t''^2}{t'^3} - \frac{t'''}{t'^2}, \\
Q_\pm[t] &= \cdots
\end{aligned}$$

We found that by setting $t_l(\tilde{u}) = t_r(\tilde{u})$, we can solve part of the gauge condition $Q_\pm = 0$. By rewriting $\varphi = \log t'$, we can reinterpret Q_0 as follows

$$0 = Q_0/N = -2t' + 2\left(\frac{t''^2}{t'^3} - \frac{t'''}{t'^2}\right) + 2\Delta\eta t'^{2\Delta-1} \tag{5-9}$$

$$= 2e^{-\varphi}\left[-\varphi'' - e^{2\varphi} + \eta\Delta e^{2\Delta\varphi}\right] \tag{5-10}$$

So the black part is the real constraint condition $Q_0 = 0$, and through observation, we can see above constraints is the EOM of the following action

$$S/N = \int d\tilde{u}\dot{\varphi}^2 - V(\varphi)\,, V = e^{2\varphi} - \eta e^{2\Delta\varphi} \tag{5-11}$$

When we consider the excitation states, we can need to modify the gauge condition on Q_0. This action eqn $5-11$ is useful in describing gauging condition for Ground State. And it is helpful in seeing the excitation behavior with analysis on $V(\varphi)$, and helpful for us to derive the behavior of energy gap.

We can see that $\sqrt{2}\varphi$ is a canonical variable, which we denote as $\tilde{\varphi}$ Then we can write the canonical form as

$$\int d\tilde{u}\left(\frac{1}{2}\dot{\tilde{\varphi}}^2 - \left(e^{\sqrt{2}\tilde{\varphi}} - \eta e^{\sqrt{2}\Delta\tilde{\varphi}}\right)\right) \tag{5-12}$$

We can find that at $\tilde{\varphi} = \tilde{\varphi}_m$ the potential $V(\tilde{\varphi})$ reaches its minimum, and we obtain

$$t' = (\eta\Delta)^{\frac{1}{2-2\Delta}} \tag{5-13}$$

Also, we have $V''(\tilde{\varphi}_m) = 2(1-\Delta)\Delta^{\frac{1}{1-\Delta}}\eta^{\frac{1}{1-\Delta}}$. Therefore, if we quantize this action, we can see that there are harmonic oscillator-like excitations around $\tilde{\varphi} \approx \tilde{\varphi}_m$

5.3 Energy Gap of the System

Since we are considering the Schwarzian region, $|TFD\rangle$ is only an approximate ground state, and the energy of the system is not actually zero. We expect to obtain a gapped phase, and we are at a position close to the ground state. Also, to facilitate the discussion of low temperature later, we need to estimate the distance to the first excited state.

Claim: For a bulk field with conformal weight Δ, the energy spectrum based on the Virasoro algebra is $E_t = \Delta + n\bullet$; while the energy spectrum obtained from the effective potential excitations in eqn $5-11$ is

$$\Delta E_{\tilde{u}} = t'\sqrt{2(1-\Delta)}, \quad E_{\tilde{u}} = \left(n + \frac{1}{2}\right)\Delta E_{\tilde{u}} \tag{5-14}$$

$$\boxed{\begin{aligned} E_{\text{conformal}} &= t'(\Delta + n) \\ E_{\text{potential}} &= t'\sqrt{2(1-\Delta)}\left(n + \frac{1}{2}\right) \end{aligned}} \tag{5-15}$$

Since we consider $q \geq 2$ in SYK, we have $\Delta = \frac{1}{q} \leq \frac{1}{2}$, so we can see that

$$E_G^{\text{poten}} = \frac{1}{2}t'\sqrt{2(1-\Delta)} \geq t'\Delta = E_G^{\text{conf}} \tag{5-16}$$

Therefore, we know that the first excited state in the conformal region we are discussing is at $E = t'\Delta$.

5.4 Energy of the Schwarzian Action

We now derive the energy corresponding to the action we have been studying, which is not a simple task because it involves higher-order derivatives. We present the derivation idea using the Ostrogradsky method. Due to time constraints, the issue of signs has not been perfectly resolved and is marked in purple.

First, we derive the energy of $\text{Sch}(f(u), u)$. Since the Schwarzian involves higher-order derivatives, we cannot use the simple Legendre transformation. The correction method corresponds to introducing the Ostrogradsky canonical variables.

Constructing Ostrogradsky Variables:

For $S = \int dt\, \mathcal{L}\left(q, \dot{q}, \ddot{q}, \ldots, q^{(n)}\right)$, the generalized coordinates should be extended as follows:

$$Q_0 = q, \quad Q_1 = \dot{q}, \quad Q_2 = \ddot{q}, \quad \cdots, \quad Q_{n-1} = q^{(n-1)}. \tag{5-17}$$

For each coordinate Q_i, define the conjugate momentum P_i:

$$P_i = \sum_{k=i+1}^{n}\left(-\frac{d}{dt}\right)^{k-i-1}\left(\frac{\partial \mathcal{L}}{\partial q^{(k)}}\right), \quad i = 0, 1, \ldots, n-1. \tag{5-18}$$

Specifically:

- When $i = n-1$,

$$P_{n-1} = \frac{\partial \mathcal{L}}{\partial q^{(n)}}. \tag{5-19}$$

- When $i < n-1$, multiple time differentiations with negative signs need to be applied to the higher-order partial derivatives.

And we'll define Ostrogradsky Hamilton as follows

$$H = \sum_{i=0}^{n-1} P_i \dot{Q}_i - L. \tag{5-20}$$

Energy of the Schwarzian Part:

$$\begin{aligned} P_3 &= \frac{\partial \mathcal{L}}{\partial f'''} = -\frac{1}{f'}, \\ P_2 &= \frac{\partial \mathcal{L}}{\partial f''} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial f'''}\right) = 2\frac{f''}{f'^2}, \\ P_1 &= \frac{\partial \mathcal{L}}{\partial f'} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial f''}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial f'''}\right) = \frac{f''^2}{f'^3} - \frac{f^{(3)}}{f'^2}. \end{aligned} \tag{5-21}$$

After substitution, the final result is (5-22)

$$-\mathcal{L} = H = \mathrm{Sch}(f(u), u). \tag{5-23}$$

However, when we obtain eqn $5-25$, we use the sign written in[6].

Energy of the Interaction Term

Consider the coupled SYK model with interactions. Since the highest-order derivative in the interaction term is only a first-order derivative, the corresponding Hamiltonian is $\sum_i p_i \dot{q}_i - \mathcal{L}_{\text{int}}$.

$$H_{\text{int}} \supset \eta(2\varDelta - 1)\left[\frac{t_l'(\tilde{u}) t_r'(\tilde{u})}{\cos\left(\frac{t_l - t_r}{2}\right)}\right]^{\varDelta} \tag{5-24}$$

5.5 Energy of the Full Action

$$\begin{aligned} \frac{E_{\tilde{u}}}{N} &= -\{\tan\frac{t_l(\tilde{u})}{2}, \tilde{u}\} - \{\tan\frac{t_r(\tilde{u})}{2}, \tilde{u}\} + \eta(2\varDelta - 1)\left[\frac{t_l'(\tilde{u}) t_r'(\tilde{u})}{\cos\frac{t_l - t_r}{2}}\right]^{\varDelta} \\ &= -(2\varphi'' - \varphi'^2 + e^{2\varphi}) - \eta(1 - 2\varDelta)e^{2\varDelta\varphi} \\ &\cong (\varphi'^2 + e^{2\varphi}) - \eta e^{2\varDelta\varphi} \\ &\cong -\frac{(1-\varDelta)}{\varDelta}(\eta\varDelta)^{\frac{1}{1-\varDelta}} \end{aligned} \tag{5-25}$$

The first step is obtained by using the variable $\varphi = \log(t')$ *and the gauge condition to simplify the expression.*

The second step uses the EOM obtained from $Q_0 = 0$: $\varphi'' + e^{2\varphi} = \eta\Delta e^{2\Delta\varphi}$.

The third step is obtained when considering the linear solution, where $\varphi'' = 0$, *and we get* $e^{2\varphi} = (\eta\Delta)^{\frac{1}{1-\Delta}}$.

5.6 After Interaction

In the GJW protocols with the interaction term containing $\theta(-u)$, this scenario corresponds to describing a decoupled system at $u > 0$. Mathematically, this corresponds to making the η term vanish. For convenience, we still use the linear solution, corresponding to φ' and higher-order derivatives all vanishing. We also need the energy to satisfy the equation of motion, i.e., $Q_0 = 0$. Specifically, solving$\hat{E}_{\tilde{u}}$ corresponds to erasing the η term in the third step above.

$$E^G_{\tilde{u}} = Nt'^2 \tag{5-26}$$

Since the linear solution on Rindler time solves the EOM of single Schwarzian, we have

$$t_l = t_r = \arctan\left(\tanh\left(\frac{\pi\tilde{u}}{\tilde{\beta}}\right)\right) \text{ or } = \arctan\left(-\frac{1}{\tanh\left(\frac{\pi\tilde{u}}{\tilde{\beta}}\right)}\right), \tag{5-27}$$

$\tilde{\beta}$ here describe the temperature after we turn off the interaction. And note that both of the expression gives a consistent solution for φ which is

$$\varphi = \log t' = \log\left[\frac{2\pi}{\tilde{\beta}\cosh\left(\frac{2\pi\tilde{u}}{\tilde{\beta}}\right)}\right] \tag{5-28}$$

This φ always satisfies $Q_0^{\eta=0} = 0$. Note that at u = 0, the crossing point for $\eta = 0$, solution t' is $(\eta\Delta)^{\frac{1}{2(1-\Delta)}}$ determined by$Q_0^{\eta\neq 0} = 0$. Therefore, we can obtain $\tilde{\beta} = \frac{1}{2\pi}(\eta\Delta)^{-\frac{1}{2(1-\Delta)}}$.

5.7 Low Temperature Limit

Lowering the temperature means that we need to wrap the time dimension. Although in Euclidean AdS_2, the condition to be satisfied is $t \sim t + \beta'$, but β' does not represent the actual temperature. The actual $\tilde{\beta}_{ph}$ is determined by $\tilde{u} \sim \tilde{u} + \tilde{\beta}_{ph}$ •. Thus, we impose the following periodicity relations and obtain the corresponding solutions, with linear solution and gauge fixing concerned.

$$\begin{aligned} t_l(\tilde{u} + \tilde{\beta}_{ph}) &= t_l(\tilde{u}) + \beta', \\ t_r(\tilde{u} + \tilde{\beta}_{ph}) &= t_r(\tilde{u}) + \beta', \\ t_l = t_r &= \frac{\beta'}{\tilde{\beta}_{ph}} \tilde{u}. \end{aligned} \tag{5-29}$$

Based on previous discussions, we obtain

$$\frac{S}{N} = \int d\tilde{u} \left(\dot{\varphi}^2 - e^{2\varphi} + \eta e^{2\Delta\varphi} \right). \tag{5-30}$$

For the linear ansatz we consider, we have

$$t' = \frac{\beta'}{\tilde{\beta}_{ph}}. \tag{5-31}$$

Introducing the contribution of the bulk action, which is determined by the global time period β', denoted as $Z_{\text{bulk}}(\beta')$. The Euclidean action is then given by

$$-\frac{S_E}{N} = \log Z_{\text{bulk}}(\beta') + \tilde{\beta}_{ph} \left[-(t')^2 + \eta (t')^{2\Delta} \right], \tag{5-32}$$

Remark

- The sign originates from wick rotation. We can see that when we are doing wick rotation, $t \longrightarrow it_E, u \longrightarrow u_E$. But we still use t, u convention in the following discussion.

$$\text{Sch}(t(u), u) \equiv \frac{t'''}{t'} - \frac{3}{2} (\frac{t''}{t'})^2 \rightarrow -\text{Sch}(t(u), u), \tag{5-33}$$

$$\frac{t'_r t'_l}{\cos^2(\frac{t_l - t_r}{2})} \rightarrow -\frac{t'_l t'_r}{\cosh^2(\frac{t_l - t_r}{2})} \tag{5-34}$$

- The integrand of $\int d\tilde{u}$ does not contain $\tilde{u}$; we directly write it as $\tilde{\beta}_{ph}$.
- The plus part has two interpretations: On one hand, it is the linearized ansatz version of action in eqn 5-11, which satisfies the gauge condition $Q_0 = 0$ when on shell. On the other hand, it is the action in 4-27 with its linear solution that kills derivative higher than one, only $\frac{t'^2}{2}$ left with $\tan(t/2)$ mapping. Therefore, we can see this part of action is of correct form that carries both physical meaning and gauge condition.

Since β' is treated as a parameter, we'll deal its variation:

$$0 = -2t' + 2\eta\Delta(t')^{2\Delta-1} - \epsilon(\beta'), \qquad \epsilon(\beta') = -\partial_{\beta'} \log Z_{\text{bulk}}. \tag{5-35}$$

We now consider the low-temperature approximation. At low temperatures, $\Delta\beta' \gg 1$•, we can use the following approximation. Since the energy of the first excited state of the system we are discussing is Δ, we can approximate

$$Z_{\text{bulk}}(\beta') = e^{-\Delta\beta'}, \tag{5-36}$$

and thus we can approximate

$$\epsilon \sim \Delta e^{-\Delta\beta'}. \tag{5-37}$$

This gives the dynamical equation in the low-temperature region:

$$0 = -2t' + 2\eta\Delta(t')^{2\Delta-1} - \Delta e^{-\Delta\beta'}, \qquad t' = \frac{\beta'}{\tilde{\beta}_{ph}}. \tag{5-38}$$

5.8 Phase Transition

The phase diagram of the MQ model features an intriguing phase transition analogous to the Hawking-Page transition, shifting from a gapped, traversable wormhole phase at low temperatures to a gapless SYK non-Fermi liquid phase at high temperatures. This first-order phase transition is marked by a significant loss of entropy , seen in figure 5-1, and can be understood from the gravity perspective as a transition from a black hole to a thermal gas at low temperatures in AdS spacetime[28].

5.9 Bulk Interpretation

In this subsection, we are discussing how to interpret the solution in Gravity Picture.

We'll slightly review the coordinate system discussed in chapter 4 before further discussion.

5.9.1 Review Coordinate system

In global coordinate syste, we use coordinate (T, σ) where $T \equiv t_{l,r}$ in MQ's notation.

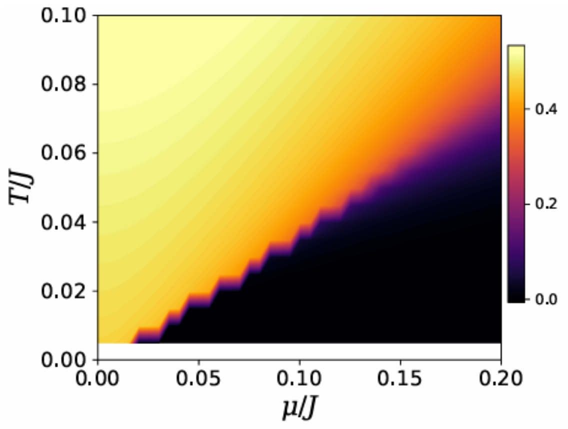

Figure 5-1 Entropy of the system[28]

$$
\begin{aligned}
Y_{-1} &= \frac{\cos(T)}{\sin(\sigma)} \\
Y_0 &= \frac{\sin(T)}{\sin(\sigma)} \\
Y_1 &= \cot(\sigma)
\end{aligned}
$$

In poincare coordinate system, we use coordinate (t_p, z), It is consistent with what we discussed in Chapter 4 with $\mu = 1$

$$
\begin{aligned}
Y_{-1} &= -\frac{1}{2}\left(\frac{1}{z} + z\right) + \frac{t_p^2}{2z}, \\
Y_0 &= -\frac{t_p}{z}, \\
Y_1 &= -\frac{1}{2}\left(\frac{1}{z} - z\right) - \frac{t_p^2}{2z}.
\end{aligned}
$$

5.9.2 Boundary of JT gravity

In chapter 3, we mentioned the boundary condition in JT gravity implies $z = \epsilon t_p'$, we'll see how to interpret this boundary,with MQ's result in eqn 5-13 that $t_l' = t_r' = \text{const}$, in gravity picture.

In Poincare coordinate, $Y_{-1} \approx \frac{t_p^2 - 1}{2z}$, and we have:

$$
Y_{-1}^2 + Y_0^2 = \left(\frac{t_p^2 + 1}{2\epsilon t_p'}\right)^2
$$

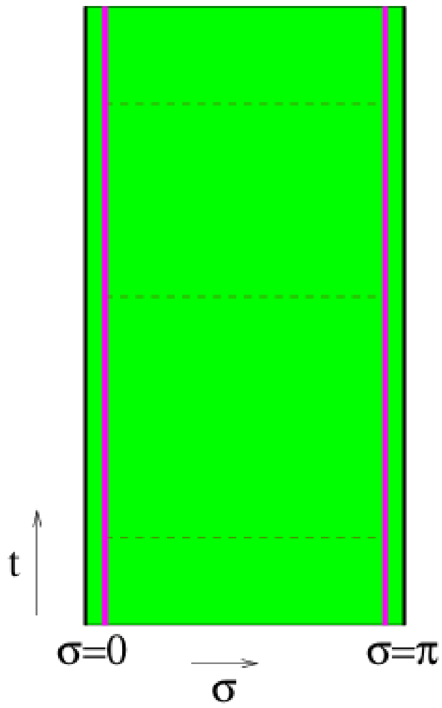

Figure 5-2 Linear Solution in Penrose Diagram (Global coordinate)[6]

In global Coordinate, we have

$$Y_{-1}^2 + Y_0^2 = \frac{1}{\sin^2 \sigma}$$

Notice that using $\frac{Y_0}{Y_1+Y_{-1}}$, we can see that $t_p = \tan(\frac{T}{2})$. Therefore, we have

$$\frac{\sec(\frac{T}{2})}{2\epsilon \sec(\frac{T}{2}) \cdot T'(u)} = \frac{1}{\sin \sigma}$$

Once we use the result $T' \equiv t_l' = t_r' = \pm$constant in MQ model's solution, we can see our result equals to $\sigma = C$ and this means the solution , representing the boundary trajectory and shown in pink line in figure 5-2, extends to the future.

We can see the boundary extend straightly to the future. To some degree, it represents a traversable, though more evidence in the energy spectrum need to be discussed before calling it a WH in a formal way. When we emit a photon from one of the bouundary, it travels along the 45 degree line and would ultimatly reach the boundary on the other side. However, in simple JT gravity, the boundary trajectory looks like what shown in figure 5-3 where no signal can be send to the other boundary trajectory.

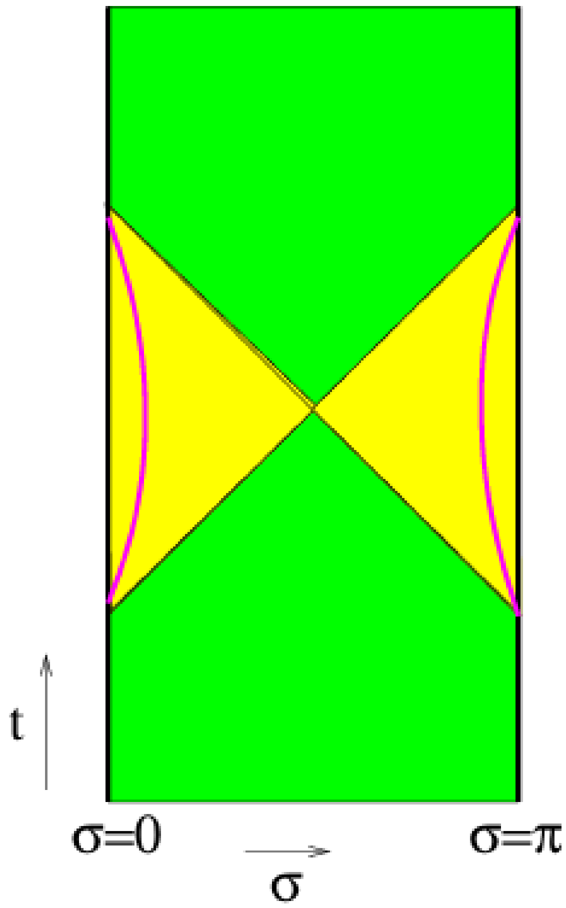

Figure 5-3 Physical boundaries of N-AdS_2

5.10 Beyond Schwarzian

5.10.1 Review MQ Setting

We have

$$\boxed{H_{\text{tot}} = H_L^{\text{SYK}} + H_R^{\text{SYK}} + H^{\text{int}}} \tag{5-39}$$

where $J^L_{i_1\cdots i_q} = (-1)^{q/2} J^R_{i_1\cdots i_q}$, $H^{\text{int}} = i\mu \sum_j \psi_L^j \psi_R^j$.

The previous discussion was directly based on $S_L^{\text{free}} \sim S_R^{\text{free}} \propto \int du\, \text{Sch}$. However, the discussion here that deviates from the Schwarzian region needs to refer to the specific techniques of Gaussian integrals in Chapter 2. In principle, we need to carefully discuss whether it is proper to use Euclidean signature or Lorentzian signature. However, we are not going to focus on them in this section, it would be lightly discussed the difference in next section. Here we'll simply follow the procedure in[6] and deriving SD eqn in coupled SYK.

5.10.2 Gaussian Integral

In the process of performing the Gaussian integral, we are effectively dealing with the following ensemble average:

$$J_{i_1\cdots i_q}(\psi_{i_1}^L\cdots\psi_{i_q}^L-(-1)^{q/2}\psi_{i_1}^R\cdots\psi_{i_q}^R) \tag{5-40}$$

The difference from what discussed in chapter 2 is that the squared term in the integral result includes

$$X_LX_L+(-1)^qX_RX_R+(-1)^{q/2}(X_LX_R+X_RX_L) \tag{5-41}$$

where $X_a=i^{q/2}\psi_{i_1}^a\cdots\psi_{i_q}^a$. This effect corresponds to the appearance of the following form in the final $G\Sigma$:

$$\log\langle Z\rangle/N=-\frac{S_E}{N}\supset\frac{1}{2}\sum_{ab}\iint\left[\frac{1}{q}J^2s_{ab}G_{ab}(\tau_1,\tau_2)^q\right] \tag{5-42}$$

where $a,b=\{L,R\}$, $s_{LL}=s_{RR}=1$, $s_{LR}=s_{RL}=(-1)^{q/2}$. Since $q\in2\mathbb{Z}$, we do not need to consider the effect of $(-1)^q$. Discussion on G_{ab} would be presented in next paragraph.

5.10.3 Path Integral and Functional Determinant

Considering the complete action, we need to introduce the kinetic term, which is specifically expressed as

$$\psi_a\partial\psi_a,\quad a=\{L,R\}. \tag{5-43}$$

Moreover, in addition to introducing G_{LL} and G_{RR}, due to the coupling of X_RX_L and X_LX_R, we must introduce G_{LR} and G_{RL}. Specifically, we need to introduce

$$\int\mathcal{D}\Sigma_{ab}(\tau_1,\tau_2)\exp\left(-\frac{N}{2}\Sigma_{ab}(\tau_1,\tau_2)\left(G_{ab}(\tau_1,\tau_2)-\frac{1}{N}\sum_{i=1}^N\psi_a^i(\tau_1)\psi_b^i(\tau_2)\right)\right). \tag{5-44}$$

Therefore, in the action, we will have terms of the form $\psi_a(\partial-\Sigma_{aa})\psi_a$ and $\psi_a\Sigma_{ab}\psi_b$, which are quite different from our previous discussions in chapter 2.

In the two-site SYK model, it is no longer as simple as single SYK. Since the inserted identity in partition function have $\psi_aA_{ab}\psi_b$ term which is no longer a binary form suited for gaussian integration. We can no longer use the standard path integral mentioned in chapter 2. However, with small modification that we write $\Psi=(\psi^+,\psi^-)$, we can see $\Psi A\Psi$ in the action with consistent form shown in standard SYK. The path integral brings us $\exp(\ln\det(A))$ after integration on Ψ. Here

$$A = \begin{pmatrix} \partial_\tau - \Sigma_{aa} & -\Sigma_{ab} \\ -\Sigma_{ba} & \partial_\tau - \Sigma_{bb} \end{pmatrix}, \tag{5-45}$$

5.10.4 Solving Coupled SD Eqn

Since the action looks greatly different, we'll new discussion on how to derive SD eqn. It would be hard to derive the SD eqn in a simple form with the technique mentioned in chapter 2. We'll following the discussion in appendix B, explicitly with eqn B-29. We can obtain

$$\begin{aligned} \delta \ln\det(\mathbb{A})_{ab} &= -\frac{\tilde{\mathbb{A}}_{ab}}{\det(A)}\delta\Sigma_{ab} \\ &= -\frac{\delta_{ab}\partial_\tau - \Sigma_{ab}}{\det(A)}\delta\Sigma_{ab}. \end{aligned} \tag{5-46}$$

The complete action and variation result are:

$$\begin{aligned} \frac{I}{N} = &-\frac{1}{2}\log\det(\delta_{ab}\partial_\tau - \Sigma_{ab}) \\ &+ \frac{1}{2}\sum_{ab}\iint\left[\Sigma_{ab}(\tau,\tau')G_{ab}(\tau,\tau') - \frac{1}{q}J^2 s_{ab}G^q_{ab}(\tau,\tau')\right]d\tau d\tau' \\ &+ \frac{i\mu}{2}\int\left[G_{LR}(\tau,\tau) - G_{RL}(\tau,\tau)\right]d\tau. \end{aligned} \tag{5-47}$$

$$\frac{\tilde{\mathbb{A}}_{ab}}{\det(A)} + G_{ab}(\tau,\tau') = 0. \tag{5-48}$$

Forming a Matrix

$$\frac{1}{\det(\mathbb{A})}\begin{pmatrix} \tilde{\mathbb{A}}_{aa} & \tilde{\mathbb{A}}_{ab} \\ \tilde{\mathbb{A}}_{ba} & \tilde{\mathbb{A}}_{bb} \end{pmatrix} + \begin{pmatrix} G_{aa}(\tau,\tau') & G_{ab}(\tau,\tau') \\ G_{ba}(\tau,\tau') & G_{bb}(\tau,\tau') \end{pmatrix} = 0. \tag{5-49}$$

Denoting

$$\begin{pmatrix} G_{aa}(\tau,\tau') & G_{ab}(\tau,\tau') \\ G_{ba}(\tau,\tau') & G_{bb}(\tau,\tau') \end{pmatrix} \tag{5-50}$$

as $\mathbb{G}$, we obtain from our previous derivation

$$(\mathbb{A}^{-1}) + \mathbb{G} = 0. \tag{5-51}$$

This implies

$$\mathbb{A}\cdot\mathbb{G} = \delta_{ab}\delta(\tau_1 - \tau_2). \tag{5-52}$$

$$\int d\tau'' \begin{pmatrix} \partial_\tau - \Sigma_{aa}(\tau,\tau'') & -\Sigma_{ab}(\tau,\tau'') \\ -\Sigma_{ba}(\tau,\tau'') & \partial_\tau - \Sigma_{bb}(\tau,\tau'') \end{pmatrix} \cdot \begin{pmatrix} G_{aa}(\tau'',\tau) & G_{ab}(\tau'',\tau) \\ G_{ba}(\tau'',\tau) & G_{bb}(\tau'',\tau) \end{pmatrix} = \delta_{ab}\delta_{\tau,\tau'}. \tag{5-53}$$

Since

$$\delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5-54}$$

substituting $a, b = L, R$ yields

$$\begin{aligned} &\partial_\tau G_{LL}(\tau) - \Sigma_{LL} * G_{LL}(\tau) - \Sigma_{LR} * G_{RL}(\tau) = \delta(\tau), \\ &\partial_\tau G_{LR} - \Sigma_{LL} * G_{LR} - \Sigma_{LR} * G_{RR} = 0. \end{aligned} \tag{5-55}$$

Chapter 6 Different Protocols In MQ model

Discussion by Maldacena and Qi on coupled SYK has been highly inspiring to this field, especially for the mathematical structure of the dynamics of coupled SYK-like actions. This chapter refers to the paper[11] to discuss the dynamics of models similar to MQ.

The first part, referring to this paper, describes the feasibility conditions for continuous measurements of two SYKs. It turns out having similar to MQ's model, but in my thesis, I would only verbally repeat the approximation method in[11] on continuous measurement. In the second part, I'll discuss the impact of instantaneous interaction on the dynamics of the MQ model.

6.1 Continuous Measurements and MQ Model

Specific measurements can cause the von Neumann entropy of a system to transition from an area law to a volume law, a phenomenon known as Measurement-induced phase transition(MIPT); similarly, there is a wormhole phase transition in the gravitational dual of coupled SYK. This section focuses on MIPT in teleportation and wormholes.

6.1.1 How to Properly Describe Measurement:

First, we need to discuss how to properly describe measurement. This paper focuses on KY protocols[29] shown in figure 6-1. Specifically, after evolving for a while, the system is projected onto the maximally entangled state of the L, R systems to complete the measurement.

Before analyzing the dynamics, we have to review the construction of the system and how to build the protocols.

6.1.2 Construction of the System

First, our system consists of $2N$ Majorana fermions, $\psi_L^i, \psi_R^i, \quad i = 1, \ldots, N$, satisfying the following conditions:

$$\left\{\psi_\alpha^i, \psi_\beta^j\right\} = \delta^{ij}\delta_{\alpha\beta}, \quad \alpha, \beta = L, R, \quad i = 1, \ldots, N \tag{6-1}$$

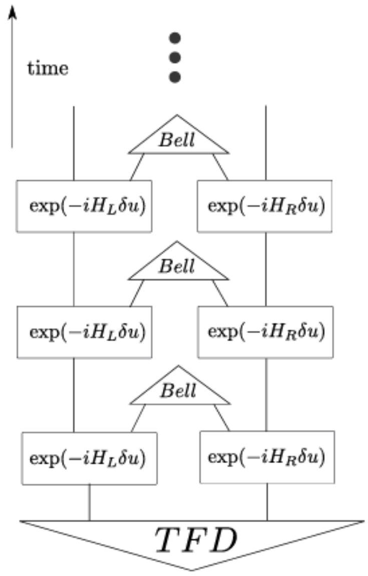

Figure 6-1 KY protocols

Remark: Though L, R fermions are anti commuted in its construction. After the KY protocols, the anti-commutation of $\psi_{L,R}$ is no longer 0.

The system we consider is in the TFD state, which is constructed above following state:

$$|TFD\rangle = \sum_{E_n} e^{-\beta E_n/2} |n_L\rangle \overline{|n_R\rangle}, \tag{6-2}$$

6.1.3 Construction of Protocols

Projection Operator

According to the KY protocols, the measurement operator projecting onto the maximally entangled state is constructed as follows:

$$\Pi = \prod_{j=1}^{N} \Pi_j = \prod_{j=1}^{N} \frac{1}{2}\left(1 - i\psi_L^j \psi_R^j\right). \tag{6-3}$$

Measurement Protocol

We aim to describe continuous measurements. Define the projection rate κ, and the corresponding measurement protocol is described as:

$$\widetilde{\rho}(u+du) = i[H,\widetilde{\rho}(u)]du+\widetilde{\rho}(u)(1-4N\kappa du)+4N\kappa du\sum_j \frac{1}{N}\cdot\frac{1}{2}(1-i\psi_L^j\psi_R^j)\widetilde{\rho}(u)\frac{1}{2}(1-i\psi_L^j\psi_R^j). \tag{6-4}$$

The corresponding physical interpretation is that within the time du, there is a probability of $1 - 4N\kappa du$ of not performing the measurement. If a measurement is performed, we randomly select $1 \le j \le N$ for the projection measurement and only retain the result with an eigenvalue of 1.

Measurement to Path Integral Representation

The Keldysh path integral is a natural path integral description of the von Neumann equation, which describes the evolution of ρ. The Schwinger-Keldysh formalism is used subsequently. The continuous measurement protocol can be obtained through the above method theoretically. But continuous measurements in QFT will introduce UV divergence[11], therefore Milekhin has to use weak projections, or measurements with post-selection, to avoid this problem.

Technically, the KY protocols under weak projection can be expressed as:

$$\boxed{\begin{aligned} iS_{\text{weak proj}} &= -\kappa\sum_j\int du\,\left(i\psi^+_{R,j}\psi^+_{L,j} + i\psi^-_{L,j}\psi^-_{R,j}\right), \\ iS_{MQ} &\propto \mu\sum_j\int du\,\left(\psi^+_{L,j}\psi^+_{R,j} - \psi^-_{L,j}\psi^-_{R,j}\right). \end{aligned}} \tag{6-5}$$

We can see the resemblance between MQ/GJW protocols and Milekhin's weak projection protocols. The dynamic of $S_{\text{weak proj}}$ wouldn't be discussed here.

6.1.4 Characterizing Information Transfer in the System

After well-estalished the protocols, a method shall be specified to detect the system. In Milekhin's paper, it is mentioned that the Teleportation fidelity is proportional to the anti-commutator for fermions lying on the L, R boundaries. Therefore, the information transfer between boundaries is characterized by $\text{Im}\,G_{LR}(u_1,u_2) = -i\text{Tr}\left(\rho_{LR}\{\psi_L(u_1),\psi_R(u_2)\}\right)$. Specifically, we will start the protocol at $u = 0$, insert information on the L side at $u = u_1$, and then detect the ability to detect the information on the R side at time T. The mathematical description is $\text{Im}G_{LR}(u_1,T)$. And deriving the two-point function would be our aim in the following discussion.

6.2 MQ Model with Instantaneous Interactions

Unlike the GJW protocols, we replace $\theta(-u) \to \delta(u - u^*)$ and observe the impact. The physical interpretation is to instantaneously open interactions, or act on a single measurement to the system. We'll determine how information transfer between tow copies of SYK . Similar to the previous discussion, correlation functions are used to discuss information transfer, but here the choice of $\tilde{G}_{LL\bullet}$ is indeed puzzling. We still follow Milekhin's discussion and present the solution approach for the Schwarzian dynamics corresponding to a single projection. Though the specific solution has not been verified yet.

As mentioned before, imaginary part of green's function is used to describe how information get transferred. One of the correlation function we are interested in correlation functions is as follows

$$\langle TFD|\psi_L(u_1)\Pi_\kappa(0)\Pi_\kappa(0)\psi_L(u_2)|TFD\rangle \tag{6-6}$$

We believe that the inserted instantaneous interaction only changes the boundary conditions of G and Σ, similar to the discussion in[7]. Therefore, theoretically, by solving for the f mode with proper boundary condition, we can solve the correlation function with the inserted projection operator in large N, low energy region.

6.3 Note for Schwarzian in MIPT

This section formally discusses the dynamics in the model proposed by Milekhin. Corresponding to the measurement at $u = u^*$, we obtain the following dynamics:

$$S = -\int d\tau \ \mathrm{Sch}(f_L, u) - \int d\tau \ \mathrm{Sch}(f_R, u) + i\kappa \int du \ \delta(u - u^*) \left(\frac{f_L'(u) f_R'(u)}{(f_L(u) - f_R(u))^2} \right)^2 \tag{6-7}$$

Now we need to solve the Equation of Motion. First, we solve the variation of the Schwarzian, then discuss the variation of the interaction, and finally obtain the EOM.

Variation of the Schwarzian Part:

$$\delta \mathrm{Sch} = -(\mathrm{Sch})' \frac{\delta f_L}{f_L'} \tag{6-8}$$

Proof:

Without loss of generality, we can consider the Schwarzian action in the IR JT action.

$$I_{sch} = -\frac{1}{8\pi G_N}\int du\ \phi(u)\mathrm{Sch}(f(u),u) \tag{6-9}$$

And we'll do the variation with respect to $f(u)$, and in the final step, we'll set $\phi = 1$.

$$\begin{aligned}
&\int \phi\left(\frac{f'f'''}{f'^2} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)\\
&= \int \phi\left(\frac{f''}{f'}\right)' - \phi\frac{1}{2}\left(\frac{f''}{f'}\right)^2\\
&= -\int \phi'\frac{f''}{f'} + \phi\frac{1}{2}\left(\frac{f''}{f'}\right)^2\\
&= \int -\left[\phi\frac{\delta f''}{f'} - \phi\frac{f''}{f'^2}\delta f' + \phi\frac{f''\delta f''}{f'^2} - \phi\frac{f''^2}{f'^3}\delta f'\right]\\
&= \int -\left[\delta f''\cdot\left(\frac{\phi'}{f'} + \phi\frac{f''}{f'^2}\right) - \delta f'\cdot\left(\phi'\frac{f''}{f'^2} + \phi\frac{f''^2}{f'^3}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \int \delta f\cdot -\left[\left(\frac{\phi'}{f'} + \phi\frac{f''}{f'^2}\right)'' + \left(\phi'\frac{f''}{f'^2} + \phi\frac{f''^2}{f'^3}\right)'\right]\\
&= -\int df\left[\left(\frac{(\phi f')'}{f'^2}\right)'' + \left(\frac{f''}{f'^3}\cdot(\phi f')'\right)'\right]\delta f
\end{aligned}$$

Setting ϕ as 1 and what in the bracket gives us

$$\frac{3(f'')^3 + f^{(4)}(f')^2 - 4f'''f'f''}{f'^4}$$

Since Sch defines as $\dfrac{f'''}{f'} - \dfrac{3(f'')^2}{2(f')^2}$, we can check $\dfrac{\mathrm{Sch}'}{f'}$ is the same.

Variation of the Interaction Part:

The interaction part and the variation result are as follows:

$$S_{\text{int}} = \kappa\int du\,\delta(u-u^*)\left(\frac{f_L'(u)f_R'(u)}{(f_L(u)-f_R(u))^2}\right)^{\Delta} \tag{6-10}$$

$$\delta S_{\text{int}} = -\frac{\delta f_L}{f_L'(u^*)}\delta'(u-u^*)\kappa\Delta\left(\frac{f_L'(u^*)f_R'(u^*)}{(f_L(u^*)-f_R(u^*))^2}\right)^{\Delta} + \frac{\mathcal{A}}{f_L'}\delta(u-u^*),\quad \mathcal{A} = \text{const.} \tag{6-11}$$

Proof:

We'll do the variation over f_L and first part is the variation over f_L', second part deals with f_L in the denominator.

$$\begin{aligned}
1: &\int du\, \delta(u-u^*)\left[\frac{f_R'}{(f_L-f_R)^2}\right]^{\varDelta} \frac{\varDelta (f_L')^{\varDelta}}{f_L'}\delta f_L' + \\
2: &\int du\, \delta(u-u^*)(f_L' f_R')^{\varDelta}\cdot \frac{-2\varDelta}{(f_L-f_R)^{2\varDelta-1}}\delta f_L \\
&\equiv A_1 \times \frac{\delta f_L}{f_L'}\delta(u-u^*)
\end{aligned}$$

Integral by part of 1:

$$\begin{aligned}
&\int du\, \varDelta \delta f_L \left\{ -\delta'(u-u^*)\frac{1}{f_L'}\left[\frac{f_L' f_R'}{(f_L-f_R)^2}\right]^{\varDelta} - \frac{1}{f_L'}\delta(u-u^*)\left[\frac{1}{f_L'}\left(\frac{f_L' f_R'}{(f_L-f_R)^2}\right)^{\varDelta}\right]' f_L' \right\} \\
&\equiv X + A_2 \times \frac{\delta f_L}{f_L'}\delta(u-u^*)
\end{aligned}$$

When we evaluate $A_1 + A_2$ at u^* we'll have $\mathcal{A}$ and putting κ back in X we'll have the result we need.

Derive the Reparametrization Mode

Since we are dealing with variation of the action with respect to f_L (Same procedure for f_R and no need for repeated discussion), we'll have $\delta \mathrm{Sch} + \delta S_{\mathrm{int}} = 0$ and when we strip the factor $\frac{\delta f_L}{f_L'}$ and put back the $i\kappa$ factor, we'll have

$$\mathrm{Sch}' = i\delta'(u-u^*)\kappa\varDelta\left[\frac{f_L' f_R'}{(f_L-f_R)^2}\right]^{\varDelta}_{u=u^*} - i\kappa\mathcal{A}\delta(u-u^*) \tag{6-12}$$

We'll use the formula of $\int \delta(x) = \theta(x)$, where $\theta(x)$ is the step function. Integrating both sides and omitting the integration constant gives us:

$$\mathrm{Sch} = i\delta(u-u^*)\kappa\varDelta\left[\frac{f_L' f_R'}{(f_L-f_R)^2}\right]^{\varDelta}_{u=u^*} - i\kappa\mathcal{A}\theta(u-u^*) \tag{6-13}$$

The Schwarzian is a function containing higher derivatives (up to order three) over $f_L(u)$.

$$\mathrm{Sch}(f(u),u) = \frac{f_L'''(u)}{f_L'(u)} - \frac{3f_L''(u)^2}{2f_L'(u)^2} \tag{6-14}$$

Therefore, the discontinuity has to be brought by $f_L'''(u)$.

$$f_L''' = i f_L'(u)\delta(u-u^*)\times\kappa\varDelta\left[\frac{f_L' f_R'}{(f_L-f_R)^2}\right]^{\varDelta}_{u=u^*} \tag{6-15}$$

We'll set all u on the right-hand side to u^* owing to the effect of the delta function and use the delta-step function relation again, we'll derive the following relation on f_L:

$$f_L''(u^*+\epsilon) - f_L''(u^*-\epsilon) = i\tilde{\kappa} f_L'(u^*), \quad \tilde{\kappa} = \kappa\Delta \left(\frac{f_L'(u^*) f_R'(u^*)}{(f_L(u^*) - f_R(u^*))^2} \right)^{\Delta}, \tag{6-16}$$

So we have obtained the effect of a single measurement on the boundary dynamics of the L side. Similarly, the same pattern applies to the R side, which means that f and f' are continuous, but f'' has a jump when $u = u^*$.

Example of Solution

This part follows Milekhin's paper[11], but there are still some places that cannot be reproduced at present, and it is not certain whether they are typos. These issues await future resolution.

Since we need to consider the Lorentzian evolution, referring to[6] and[21], we need to fix the gauge for $t_{L,R}$. Here, we provide the relationship between the Euclidean Poincare time f^β and the Global time $t_{L,R}$ as follows, to describe the f^β ansatz that satisfies the gauge condition. Then, we obtain the dynamical mode $f_{L,R}$ we need to solve through analytic continuation.

$$\tan\frac{t_L^{\pm}(u)}{2} = f_{\pm}^{\beta}(u), \quad \tan\frac{t_R^{\pm}(u)}{2} = -\frac{1}{f_{\pm}^{\beta}(-i\beta/2 - u)}. \bullet \tag{6-17}$$

To satisfy the gauge condition $Q_\pm = 0$ or the $L \leftrightarrow R$ symmetry, we need to require $t_L(u) = t_R(u)$. It can be verified that the following ansatz directly satisfies this condition:

$$f_{\pm}^{\beta}(u) = \pm\frac{e^{-\alpha u} - iA_{\pm}e^{\pm i\alpha\beta/4}}{A_{\pm}e^{-\alpha u} + ie^{\pm i\alpha\beta/4}} \tag{6-18}$$

Subsequently, we assign $f_{\pm}^{\beta}$ to the Poincare time on the Euclidean Keldysh contour $\pm$ respectively $\bullet$.

When we consider $\langle TFD|\psi_L(u_1)\Pi_\kappa(u^*)\Pi_\kappa(u^*)\psi_L(u_2)|TFD\rangle$, it is equivalent to continuously inserting two sets of measurements in the Euclidean part of the Keldysh contour. See figure 6-2 where two pairs of dots inserted in the bottom of SK contour represent measurement insertion. We can see twice there are twice the jump between f_+^β and f_-^β

Thus, we obtain a double jump in $f^{\beta''}$, and a jump from the Keldysh + contour to the Keldysh - contour. The corresponding boundary conditions are as follows:

$$f_+^{\beta}(u^*) = f_-^{\beta}(u^*), \quad f_+^{\prime\beta}(u^*) = f_-^{\prime\beta}(u^*) \tag{6-19}$$

$$f_+^{\prime\prime\beta}(u^*) = f_-^{\prime\prime\beta}(u^*) - 2i\tilde{\kappa} f_+^{\prime\beta}(u^*). \tag{6-20}$$

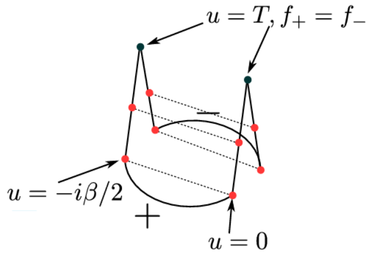

Figure 6-2　SK path integral

We choose to perform the measurement at $u^* = 0$. To facilitate the analytic continuation to Lorentzian time, it is required that $f_\pm^\beta(0) = 0$. As pointed out in[11], A_+ is a gauge choice. In conjunction with the measurement time we have selected, we are required to choose $A_\pm = -ie^{\mp i\alpha\beta/4}$ here. Consequently, we obtain the corresponding $f_\pm^\beta$:

$$f_+^\beta(u) = i\frac{\sinh\left(\frac{\alpha u}{2}\right)}{\sinh\left(\frac{\alpha u}{2} + i\frac{\alpha\beta}{4}\right)}, \tag{6-21}$$

$$f_-^\beta(u) = -i\frac{\sinh\left(\frac{\alpha u}{2}\right)}{\sinh\left(\frac{\alpha u}{2} - i\frac{\alpha\beta}{4}\right)}. \tag{6-22}$$

$$\tag{6-23}$$

It is assumed here that the information from the measurement has been decoded into f. Therefore, when we observe $\langle TFD|\psi_L(u_1)\Pi_\kappa(0)\Pi_\kappa(0)\psi_L(u_2)|TFD\rangle$ to describe the information transfer between two boundaries, we simply need to substitute the corresponding two-point function and obtain the result:

$$\left(\frac{\alpha^2/4}{-i\sinh(\alpha(u_1-u_2)/2) + \frac{2\tilde{\kappa}}{\alpha}\sinh(\alpha u_1/2)\sinh(\alpha u_2/2)}\right)^{\Delta} \tag{6-24}$$

Chapter 7 Keldysh SYK

In this chapter, we explore a model that bears a striking mathematical resemblance to the MQ model, yet its starting point is entirely different. Here, we discuss the evolution of an open SYK system, described using the Lindbladian formalism. However, we simplify the complex description through the Choi-Jamiolkowski Isomorphism, obtaining a dynamical evolution that shares significant similarities with the Euclidean two-site SYK model discussed in \ref{SUPERGS}. This chapter will first review the basic description of open system dynamics, then discuss how the Choi-Jamiolkowski Isomorphism performs vectorization operations, and finally examine two different m

7.1 From Schrödinger equation to Liouvillian equation

We are familiar with the Schrödinger equation, which can effectively describe the evolution of state vectors:

$$\frac{d}{dt}|\varphi(t)\rangle = -iH|\varphi(t)\rangle \tag{7-1}$$

If we switch to another perspective (the Heisenberg Picture), the evolution is not of $|\varphi\rangle$, but of the operators $\hat{A}(t)$ we are studying. We then obtain the Heisenberg equation:

$$\frac{d}{dt}\hat{A}(t) = i[\hat{H}, \hat{A}(t)] + \frac{\partial \hat{A}(t)}{\partial t} \tag{7-2}$$

However, the world is not composed solely of pure states. In other words, we cannot always describe a system using the simple $|\varphi\rangle$. Still in the Schrödinger picture, for mixed states, we need to consider ρ and use the von Neumann equation to describe the dynamics of the system:

$$\frac{d}{dt}\rho(t) = -i[\hat{H}(t), \rho(t)] \tag{7-3}$$

A naive description of the free evolution of ρ in the Schrödinger picture is given by (7-5). We can consider the evolution both backward and forward in time. This can be described using two copies of ψ, denoted as $\psi_\pm$, which correspond to the evolution in both directions. This approach allows us to use the Keldysh formalism to describe the real-time evolution of ρ in an open system which would be useful for later discussion but not much in current scenario.

$$\rho \to e^{-iHu} \rho e^{+iHu} \tag{7-4}$$

$$\rho \to \rho - iHdu\rho + i\rho Hdu \tag{7-5}$$

For an open system with dissipation, the evolution of ρ should be described by a differential equation using the Liouvillian. The specific form is given below, where L_i represents the Lindblad operators.

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) \tag{7-6}$$

$$\mathcal{L}(\rho) = -i[H^{\text{SYK}}, \rho] + \sum_i \gamma_i \left[L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right] \tag{7-7}$$

The purpose of this paper is not to provide a detailed and rigorous discussion of the Lindblad equation and the evolution of open systems. For more detailed discussions, we refer to[10]. The focus of this section is to introduce the use of this formalism to describe the SYK system. This formalism shares a high degree of similarity with the MQ model, and similar techniques can be applied. The specific content is mainly based on[8].

7.2 Choi-Jamiolkowski Isomorphism Vectorization

Since $\mathcal{L}(\rho)$ is a relatively complex operator, transforming $\rho \Rightarrow |\rho\rangle$ and $\mathcal{L}(\rho) \Rightarrow \mathcal{L}|\rho\rangle$ allows us to obtain a form that is very familiar in basic quantum mechanics. This process is Choi-Jamiolkowski Isomorphism (C-J Iso). For example, the evolution of $\rho(t)$ can be written as:

$$|\rho(t)\rangle = e^{\mathcal{L}t}|\rho(0)\rangle \tag{7-8}$$

Formally, this can be understood as $\mathcal{L} \cong -iH_{\text{int}}$. Following the Keldysh formalism, which naturally describes the system, we obtain two fields: $\psi_\pm$, which describe the time evolution in the positive and negative directions, respectively. The paper[8][27]provides a detailed discussion on how to describe $\mathcal{L}$ after the C-J Isomorphism:

$$\mathcal{L} = -iH_L^{\text{SYK}} + i(-1)^{\frac{q}{2}} H_R^{\text{SYK}} - i\mu \sum_i \psi_L^i \psi_R^i - \frac{1}{2}\mu N \tag{7-9}$$

One of the important details is as follows,[14]

Intuition on C-J Iso

This iso corresponds to the following mapping:

$$|i\rangle\langle j| \Rightarrow |i\rangle \otimes |j\rangle \tag{7-10}$$

According to the discussion in[14], γ can be used as a representation for $\psi_{\pm}$. Naively, we have $\gamma_i \otimes \mathbf{1}$ and $\mathbf{1} \otimes \gamma_j$, but

$$\gamma_i \otimes \mathbf{1} \cdot \mathbf{1} \otimes \gamma_j = \mathbf{1} \otimes \gamma_j \cdot \gamma_i \otimes \mathbf{1} = \gamma_i \otimes \gamma_j \tag{7-11}$$

This description loses the anti commutation property of ψ. To address this, we need to transform $\mathbf{1} \otimes \gamma_j \Rightarrow \gamma_c \otimes \gamma_j$. Detailed corrections to the construction of the TFD state under this representation are discussed in[8][27], and the following operator reflections are obtained:

$$\psi_L^k|0\rangle = -i\psi_R^k|0\rangle \tag{7-12}$$

$$|0\rangle = \sum_j |j\rangle e^{\pi i\gamma_c/4} CK|j\rangle. \tag{7-13}$$

7.3 Constructing the Lagrangian

We know that the Lagrangian can be constructed through the Legendre transformation of the Hamiltonian:

$$L = \text{Kinetic} - H \tag{7-14}$$

A naive idea is to consider the system as consisting of two independent sets of fields, $\psi_{\pm}$, so that

$$\text{Kinetic} = \frac{i}{2}\sum \psi_+\partial_t\psi_+ + \frac{i}{2}\sum \psi_-\partial_t\psi \tag{7-15}$$

However, a more rigorous discussion is needed[10]. The action takes the form

$$S = \int dt \left(\frac{i}{2}\psi_+\partial_t\psi_+ - \frac{i}{2}\psi_-\partial_t\psi_- - i\mathcal{L}(\psi_+, \psi_-)\right) \tag{7-16}$$

The negative sign in $\psi_-\partial\psi_-$ originates from the Keldysh contour C_-, which integrates in the reverse time direction. However, according to the discussion of the C-J Iso, ψ is mapped to C_- through a non-trivial mapping. Corresponding to $\psi \Rightarrow \alpha\psi_-$, it was proven in[8] that $\alpha = -i$, so our result returns to eqn 7-15.

In summary, we obtain L_{keldysh} and L_{MQ}, and we can observe a high degree of similarity in their forms.

$$L_{\text{keldysh}} = \frac{i}{2}\sum \psi_+\partial_t\psi_+ + \frac{i}{2}\sum \psi_-\partial_t\psi_- + iH^+_{\text{SYK}} - i(-1)^{\frac{g}{2}}H^-_{\text{SYK}} + i\mu\sum_i \psi^i_L\psi^i_R + \frac{1}{2}\mu N \tag{7-17}$$

$$L_{\text{MQ}} = \frac{i}{2}\sum \psi_+\partial_\tau\psi_+ + \frac{i}{2}\sum \psi_-\partial_\tau\psi_- - H^L_{\text{SYK}} - (-1)^{\frac{q}{2}}H^R_{\text{SYK}} - i\mu\sum_i \psi^i_L\psi^i_R \tag{7-18}$$

Remark:

1. The gray part does not have a dynamic mode, so it can be ignored; 2. The interaction part of MQ is Hermitian, while the interaction part of Keldysh is anti-Hermitian, so the description is different.

7.4 $G\Sigma$ Action for Liouvillian

The derivation can be conducted in different signature. Though Lorentzian signature and Euclidean signature have some significant differences in details. Lorentzian is more direct though messy in its factor. Euclidean one needs some trick and easier to conduct. Two approach are equivalent though not proven here.

7.4.1 Lorentzian signature

$$Z = \int \mathcal{D}\psi_+\mathcal{D}\psi_- e^{+iS[\psi_+,\psi_-]}, \tag{7-19}$$

We notice that the exponential factor has a plus sign. This will have a corresponding effect on the identity we insert. However, since $G_{ab}(t,t') = -\frac{i}{N}\sum_{i=1}^{N}\psi_a(t)\psi_b(t')$, it will produce a corresponding sign correction. The detailed derivation process has been shown in[8] and[30]. However, the calculations are very complicated due to the complex indices, making it difficult to solve the Schwinger-Dyson (SD) equations. Here, we use the method of Antonio and Jie

Ping to simplify the calculation. We realize that the following notation can greatly simplify the complexity of $G\Sigma$.

$$
\begin{aligned}
\psi^i(t^+) &= \psi_L^i(t)\text{with } t^+ \in C^+ \\
\psi^i(t^-) &= i\psi_R^i(t),\text{with } t^- \in C^- \\
dz &= dt,\text{with } t^+ \in C^+ \\
dz' &= -dt,\text{with } t'^+ \in C^-
\end{aligned}
$$

We can obtain the $G\Sigma$ action in the form of single SYK:

$$
iS = \frac{N}{2}\left\{\mathrm{Tr}\log(i\partial_z - \Sigma) - \int_C dz\,dz'\,\Sigma(z,z')G(z,z') - \frac{i^q J^2}{q}\int_C dz\,dz'\,[G(z,z')]^q \right. \tag{7-20}
$$

$$
\left. + 2i\mu\int_C dz\,dz'\,K(z,z')G(z,z')\right\}. \tag{7-21}
$$

It is easy to write down the SD equations. We just need to place z, z' on the appropriate contour and properly handle the sign problem to restore the SD equations for G_{ab} with $a, b \in \{+, -\}$.

$$
(i\partial - \Sigma)\cdot G = \mathbf{1}_C, \tag{7-22}
$$

$$
\Sigma(z,z') = -i^q J^2 [G(z,z')]^{q-1} + i\mu [K(z,z') - K(z',z)]. \tag{7-23}
$$

7.4.2 Euclidean signature

$$
Z = \int \mathcal{D}\psi_L \mathcal{D}\psi_R e^{-I[\psi_L,\psi_R]} \tag{7-24}
$$

The Liouvillian is defined in real time, so we need to analogize to obtain the Euclidean time evolution description of the SYK system. Specifically, by giving the Lagrangian of the system in Euclidean space, we obtain the complete description of the system.

First, we observe that the time evolution of our system is $\exp(\mathcal{L}t)$. If we describe it using Euclidean time, corresponding to $t \longrightarrow \tau$, then the evolution operator can be written as $\exp(-H\tau)$. Therefore, we can directly write the Hamiltonian of the Euclidean time system we describe:

$$H = -\mathcal{L} \tag{7-25}$$

We relabel $\psi_\pm$ as $\psi_{L,R}$, representing the two-site SYK model. Thus, we obtain an evolution equation that is consistent with the Liouvillian. Physically, it describes a non-Hermitian but PT-symmetric two-site SYK model. This process can be well compared with Chapter 5, since the derivation of the MQ model is also carried out in Euclidean signature.

7.4.2.1 Comparison of MQ and Liouvillian in Euclidean time

$$H_{MQ} = H_L^{\text{SYK}} + H_R^{\text{SYK}} + i\mu \sum_j \psi_L^j \psi_R^j$$
$$H_{\mathcal{L}} = iH_L^{\text{SYK}} - iH_R^{\text{SYK}} + i\mu \sum_i \psi_L^i \psi_R^i$$

The notation here is different from that in eqn 7-9. Here, $(-1)^{\frac{q}{2}}$ is combined into the $J_{\dots}$ coupling. The rest follows the MQ model and the Euclidean path integral, that is, introducing the kinetic term $\frac{1}{2} \sum_{a=\{L,R\},i} \psi_a^i \partial_\tau \psi_a^i$ as the result of legendre transformation shown in lagrangian. However, we should pay more attention on following difference:

- It can be seen that only the $J_{\dots}$ part is affected. Due to the effect of the i factor, the integration result $J^2 \to -J^2$.
- Since $H_L + H_R \Rightarrow H_L - H_R$, a minus sign will be generated for G_{ab} due to the different ab indices.

Therefore, compared with result in 5-47, we introduce the factor $-t_{ab}$ here to correct it in liouuvillian case, with

$$\frac{I}{N} = -\frac{1}{2} \log \det(\delta_{ab}\partial_\tau - \Sigma_{ab}) \tag{7-26}$$
$$+ \frac{1}{2} \sum_{ab} \iint \left[\Sigma_{ab}(\tau,\tau')G_{ab}(\tau,\tau') + \frac{1}{q} t_{ab} J^2 s_{ab} G_{ab}^q(\tau,\tau') \right] d\tau d\tau' \tag{7-27}$$
$$+ \frac{i\mu}{2} \int \left[G_{LR}(\tau,\tau) - G_{RL}(\tau,\tau) \right] d\tau \tag{7-28}$$
$$\text{with } t_{ab} = \begin{cases} 1 & \text{if } a = b \\ -1 & \text{if } a \neq b \end{cases} \tag{7-29}$$

We can further verify the equivalence between two approaches[8]. Though the proof is not going to be reviewed here.

Bibliography

[1] AHARONY O, GUBSER S S, MALDACENA J M, et al. Large N field theories, string theory and gravity[J]. Phys. Rept., 2000, 323: 183-386. arXiv: hep-th/9905111. DOI: 10.1016/S0370-1573(99)00083-6.

[2] MALDACENA J. International Journal of Theoretical Physics, 1999, 38(4): 1113-1133. http://dx.doi.org/10.1023/A:1026654312961. DOI: 10.1023/a:1026654312961.

[3] GUBSER S, KLEBANOV I, POLYAKOV A. Gauge theory correlators from non-critical string theory[J/OL]. Physics Letters B, 1998, 428(1–2): 105-114. http://dx.doi.org/10.1016/S0370-2693(98)00377-3. DOI: 10.1016/s0370-2693(98)00377-3.

[4] MALDACENA J, STANFORD D. Remarks on the Sachdev-Ye-Kitaev model[J/OL]. Physical Review D, 2016, 94(10). http://dx.doi.org/10.1103/PhysRevD.94.106002. DOI: 10.1103/physrevd.94.106002.

[5] KITAEV A. A simple model of quantum holography[Z]. http://online.kitp.ucsb.edu/online/entangled15/kitaev/. Talks at KITP, April 7, 2015 and May 27, 2015. 2015.

[6] MALDACENA J, QI X L. Eternal traversable wormhole[EB/OL]. 2018. https://arxiv.org/abs/1804.00491. arXiv: 1804.00491 [hep-th].

[7] GAO P, JAFFERIS D L. A Traversable Wormhole Teleportation Protocol in the SYK Model [EB/OL]. 2021. https://arxiv.org/abs/1911.07416. arXiv: 1911.07416 [hep-th].

[8] GARCÍA-GARCÍA A M, SÁ L, VERBAARSCHOT J J M, et al. Keldysh wormholes and anomalous relaxation in the dissipative Sachdev-Ye-Kitaev model[J/OL]. Physical Review D, 2023, 107(10). http://dx.doi.org/10.1103/PhysRevD.107.106006. DOI: 10.1103/physrevd.107.106006.

[9] JAFFERIS D, ZLOKAPA A, LYKKEN J D, et al. Traversable wormhole dynamics on a quantum processor[J]. Nature, 2022, 612(7938): 51-55.

[10] SIEBERER L M, BUCHHOLD M, DIEHL S. Keldysh field theory for driven open quantum systems[J/OL]. Reports on Progress in Physics, 2016, 79(9): 096001. http://dx.doi.org/10.1088/0034-4885/79/9/096001. DOI: 10.1088/0034-4885/79/9/096001.

[11] MILEKHIN A, POPOV F K. Measurement-induced phase transition in teleportation and wormholes[J/OL]. SciPost Physics, 2024, 17(1). http://dx.doi.org/10.21468/SciPostPhys.17.1.020. DOI: 10.21468/scipostphys.17.1.020.

[12] 张鹏飞. SYK(类) 模型简介 (张鹏飞)-1[Z]. https://www.bilibili.com/video/BV1M1suehEW7/. Accessed: 2025-05-13.

[13] SACHDEV S, YE J. Gapless spin-fluid ground state in a random quantum Heisenberg magnet [J/OL]. Physical Review Letters, 1993, 70(21): 3339-3342. http://dx.doi.org/10.1103/PhysRevLett.70.3339. DOI: 10.1103/physrevlett.70.3339.

[14] SAROSI G. AdS_2 holography and the SYK model[C/OL]//Modave2017: Proceedings of XIII Modave Summer School in Mathematical Physics —PoS(Modave2017). Sissa Medialab, 2018. http://dx.doi.org/10.22323/1.323.0001. DOI: 10.22323/1.323.0001.

[15] ROSENHAUS V. An introduction to the SYK model[J/OL]. Journal of Physics A: Mathematical and Theoretical, 2019, 52(32): 323001. http://dx.doi.org/10.1088/1751-8121/ab2ce1. DOI: 10.1088/1751-8121/ab2ce1.

[16] KITAEV A, SUH S J. The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual[J/OL]. Journal of High Energy Physics, 2018, 2018(5). http://dx.doi.org/10.1007/JHEP05(2018)183. DOI: 10.1007/jhep05(2018)183.

[17] Wikipedia contributors. Berezin integral — Wikipedia, The Free Encyclopedia[EB/OL]. 2025. https://en.wikipedia.org/wiki/Berezin_integral.

[18] JEVICKI A, SUZUKI K, YOON J. Bi-local holography in the SYK model[J/OL]. Journal of High Energy Physics, 2016, 2016(7). http://dx.doi.org/10.1007/JHEP07(2016)007. DOI: 10.1007/jhep07(2016)007.

[19] JEVICKI A, SUZUKI K. Bi-Local Holography in the SYK Model: Perturbations[EB/OL]. 2016. https://arxiv.org/abs/1608.07567. arXiv: 1608.07567 [hep-th].

[20] SHEOREY S. Deformed SYK & Holography[D]. King's College London, 2024.

[21] MALDACENA J, STANFORD D, YANG Z. Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space[EB/OL]. 2016. https://arxiv.org/abs/1606.01857. arXiv: 1606.01857 [hep-th].

[22] HARTMAN T. Lectures on Quantum Gravity and Black Holes[Z]. http://www.hartmanhep.net/topics2015/gravity-lectures.pdf. Accessed on 2025-05-08. 2015.

[23] GAO P, JAFFERIS D L, WALL A C. Traversable wormholes via a double trace deformation[J/OL]. Journal of High Energy Physics, 2017, 2017(12). http://dx.doi.org/10.1007/JHEP12(2017)151. DOI: 10.1007/jhep12(2017)151.

[24] HOTTA M. Asymptotic Isometry and Two Dimensional Anti-de Sitter Gravity[EB/OL]. 1998. https://arxiv.org/abs/gr-qc/9809035. arXiv: gr-qc/9809035 [gr-qc].

[25] KITAEV A, SUH S J. Statistical mechanics of a two-dimensional black hole[J/OL]. Journal of High Energy Physics, 2019, 2019(5). http://dx.doi.org/10.1007/JHEP05(2019)198. DOI: 10.1007/jhep05(2019)198.

[26] MALDACENA J. Eternal black holes in anti-de Sitter[J/OL]. Journal of High Energy Physics, 2003, 2003(04): 021-021. http://dx.doi.org/10.1088/1126-6708/2003/04/021. DOI: 10.1088/1126-6708/2003/04/021.

[27] GARCÍA-GARCÍA A M, NOSAKA T, ROSA D, et al. Quantum chaos transition in a two-site Sachdev-Ye-Kitaev model dual to an eternal traversable wormhole[J/OL]. Physical Review D, 2019, 100(2). http://dx.doi.org/10.1103/PhysRevD.100.026002. DOI: 10.1103/physrevd.100.026002.

[28] LANTAGNE-HURTUBISE É. Holographic quantum matter : toy models and physical platforms [D]. British Columbia U., 2021. DOI: 10.14288/1.0400095.

[29] YOSHIDA B, KITAEV A. Efficient decoding for the Hayden-Preskill protocol[EB/OL]. 2017. https://arxiv.org/abs/1710.03363. arXiv: 1710.03363 [hep-th].

[30] KULKARNI A, NUMASAWA T, RYU S. Lindbladian dynamics of the Sachdev-Ye-Kitaev model [J]. Phys. Rev. B, 2022, 106(7): 075138. arXiv: 2112.13489 [cond-mat.stat-mech]. DOI: 10.1103/PhysRevB.106.075138.

[31] ZEE A. Quantum field theory in a nutshell: vol. 7[M]. Princeton university press, 2010.

[32] MURUGAN J, STANFORD D, WITTEN E. More on supersymmetric and 2d analogs of the SYK model[J/OL]. Journal of High Energy Physics, 2017, 2017(8). http://dx.doi.org/10.1007/JHEP08(2017)146. DOI: 10.1007/jhep08(2017)146.

[33] 知乎用户. 量子力学中的高斯积分[Z]. https://zhuanlan.zhihu.com/p/680456212. 访问日期: 2025-05-07. 2025.

[34] CAO S, GE X H, RUI Y C. Thermodynamic and Real-time Dynamic Properties of Complex Sachdev-Ye-Kitaev Model[EB/OL]. 2024. https://arxiv.org/abs/2103.16270. arXiv: 2103.16270 [hep-th].

Appendix A More On SYK Standard Techniques

The following discussion has no relation to our research on the MQ wormhole and can be skipped directly. However, it is very helpful for deeper understanding the derivation of the $G\Sigma$ actions and the SD equations. I'll explains the differences in functional determinant integrals for bosonic and fermionic fields, the commutativity and anticommutativity of propagators, and why the inserted $G\Sigma$ identity is chosen as $\frac{N}{2}$. It would be helpful in discussing the SYK* model for both real scalar fields and complex scalar fields in the future.

A.1 Real Scalar Bosons

Compared to the case of Majorana fermion fields, Real Scalar Bosons behave slightly different.

1: Since the kinetic term of the scalar field is $\frac{1}{2}(\partial\phi)^2$, when rewriting it as a quadratic form, there will be a negative sign brought by integration by parts, i.e. $\partial_\tau - \Sigma \Rightarrow -\partial^2 - \Sigma$.

2: Referring to the discussion of Gaussian integrals in A. Zee's book[31], we will have different signs when calculating the functional determinant.

However, since G^ϕ does not have the property of anti-commutation, the form of the Schwinger-Dyson equations is almost the same as that for Majorana fermions.

A.1.1 Derivation of Functional Determinant

We know that Gaussian integral is $\int_{-\infty}^{+\infty} dx\, e^{-ax^2+Jx} = \sqrt{\frac{\pi}{a}} e^{\frac{J^2}{4a}}$. Real scalar with infinite-dimensional gaussian integral will have the form of

$$\begin{aligned}
&\int d^D x\, e^{-x\cdot A\cdot x+J\cdot x} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1 dx_2\cdots dx_N\, e^{-x\cdot A\cdot x+J\cdot x} \\
&= \left(\frac{(\pi)^N}{\det[A]}\right)^{\frac{1}{2}} e^{\frac{1}{2}J\cdot A^{-1}\cdot J}
\end{aligned} \tag{A-1}$$

Derivation:

1. Diagonalize to obtain the quadratic form $A = O^{-1}\cdot D\cdot O$, and introduce a new set of

variables $y_i = O_{ij}x_j$. Thus, $x \cdot A \cdot x = y \cdot D \cdot y$, and $J \cdot x = (OJ) \cdot y$.

Remark: For convenience, we do not distinguish between left and right multiplication here, $J \cdot x = J^T \cdot x = J^T \cdot O^T \cdot O \cdot x = (OJ) \cdot x$.

2. The Jacobian factor is $\det O = 1$, where O is an orthogonal transformation.

3. The original expression becomes

$$\begin{aligned}\prod_i \int_{-\infty}^{+\infty} dy_i\, e^{-D_{ii}y_i^2+(OJ)_i y_i} &= \prod_i \sqrt{\frac{\pi}{D_{ii}}} e^{\frac{(OJ)_i^2}{4D_{ii}}} \\ &= \sqrt{\frac{\pi^N}{\det A}} e^{\frac{1}{4}\prod_i (OJ)_i \cdot D_{ii}^{-1} \cdot (OJ)_i} \\ &= \sqrt{\frac{\pi^N}{\det A}} e^{\frac{1}{4} J \cdot A^{-1} \cdot J}\end{aligned} \tag{A-2}$$

where the transformation from the second step to the third step is as follows:

$$\begin{aligned}\prod_i (OJ)_i \cdot D_{ii}^{-1} \cdot (OJ)_i &= (OJ) \cdot D^{-1} \cdot (OJ) \\ &= J^T \cdot O^{-1} D^{-1} O \cdot J \\ &= J \cdot A^{-1} \cdot J\end{aligned} \tag{A-3}$$

A.1.2 Real Scalar $G\Sigma$ Action and First Set of SD Equations

According to previous discussion, when the action is of the form $S[\phi] = \frac{1}{2}\int d^d x\, \phi(x)\hat{O}\phi(x)$, the path integral result would be:

$$Z = \int \mathcal{D}\phi\, e^{-S[\phi]} \propto \left(\det \hat{O}\right)^{-1/2}. \tag{A-4}$$

We can ignore the constant part since it does not affect the subsequent discussion of the equations of motion. The rest of the derivation can be referred to the discussion of the Majorana ψ field, so we can directly obtain

$$e^{-\frac{N}{2}\mathrm{Tr}\ln(-\partial_\tau^2 - \Sigma)} \subset Z(J), \tag{A-5}$$

$$\frac{1}{2}\mathrm{Tr}\ln(-\delta(\tau_1,\tau_2)\partial_{\tau_1}^2 - \Sigma) \subset I, \tag{A-6}$$

$$\frac{1}{2}\mathrm{Tr}\ln(-\delta(\tau_1,\tau_2)\partial_{\tau_2}^2 - \Sigma) + \frac{1}{2}\iint d\tau_1 d\tau_2\, \Sigma(\tau_1,\tau_2) G^\phi(\tau_1,\tau_2) \subset I. \tag{A-7}$$

Similar to the variation process for the fermionic field, but carefully handle the symmetry of G^ϕ and the corresponding signs.

$$
\begin{aligned}
&\frac{1}{2}\iint d\tau_1 d\tau_2\, \delta\Sigma(\tau_1,\tau_2)\cdot\left(-(-\partial^2-\Sigma)^{-1}(\tau_2,\tau_1)+G^{\phi}(\tau_1,\tau_2)\right)\\
=&\cdots\left(\delta\Sigma(\tau_1,\tau_2)\cdot\left(-(-\partial^2-\Sigma)^{-1}(\tau_2,\tau_1)+G^{\phi}(\tau_2,\tau_1)\right)\right)
\end{aligned}
\tag{A-8}
$$

Thus, we obtain Witten's result[32]:

$$
G^{\phi}\cong(-\partial^2-\Sigma)^{-1} \tag{A-9}
$$

A.2 Complex Scalar Bosons

A.2.1 Derivation of Functional Determinant

When Majorana variables or real variables become complex variables, things become different. First, we consider the Gaussian integral of complex variables:

$$
\begin{aligned}
&\int dz_1 dz_1^{*}\ldots dz_D dz_D^{*} e^{-z^{\dagger}Az}\\
=&\int d\tilde{z}_1 d\tilde{z}_1^{*}\ldots d\tilde{z}_D d\tilde{z}_D^{*} e^{-\sum_i \alpha_i|\tilde{z}_i|^2}\\
=&\prod_{i=1}^{D}\int d\tilde{z}_i d\tilde{z}_i^{*} e^{-\alpha_i|\tilde{z}_i|^2}
\end{aligned}
\tag{A-10}
$$

Note that the significant difference here is the existence of $\int dz$ and $\int dz^*$, which can be understood as twice the degree of integration freedom. The first step to the second step is to use an orthogonal transformation to diagonalize A, which is the same as the integration process of real Gaussian variables.

$$
\begin{aligned}
&\int \mathrm{d}\tilde{z}\mathrm{d}\tilde{z}^{*} e^{-\alpha|\tilde{z}|^2}=2\int \mathrm{d}\,\mathrm{Re}\,\tilde{z}\mathrm{d}\,\mathrm{Im}\,\tilde{z}\; e^{-\alpha(\mathrm{Re}\,\tilde{z})^2}e^{-\alpha(\mathrm{Im}\,\tilde{z})^2}\\
=&2\left(\int_{-\infty}^{\infty}\mathrm{d}x e^{-\alpha x^2}\right)^2=2\frac{\pi}{\alpha}
\end{aligned}
\tag{A-11}
$$

The factor of 2 comes from $z=x+\mathrm{i}y,\quad z^*=x-\mathrm{i}y,\quad |\mathrm{d}z\,\mathrm{d}z^*|=2\,\mathrm{d}x\,\mathrm{d}y$.

So the complete integral contribution is

$$
\int \mathrm{d}z_1\mathrm{d}z_1^{*}\ldots\mathrm{d}z_D\mathrm{d}z_D^{*}\, e^{-z^{\dagger}Az}=\prod_{i=1}^{D}\frac{2\pi}{\alpha_i}=\frac{(2\pi)^D}{\det A}=(2\pi)^D e^{-\mathrm{Tr}\ln A} \tag{A-12}
$$

We can see that the exponential index is twice larger than the real scalar case.

We can also extend our case to complex fermions, but here we directly give the extension's result with detailed discussions can be found in[33].

$$\int d\mu(\xi) e^{-\sum_{\alpha\beta} \xi^*_\alpha G_{\alpha\beta} \xi_\beta + \sum_\alpha (\eta^*_\alpha \xi_\alpha + \eta_\alpha \xi^*_\alpha)} = [\det G]^{-s} e^{\sum_{\alpha\beta} \eta^*_\alpha G^{-1}_{\alpha\beta} \eta_\beta} \tag{A-13}$$

$$d\mu(\xi) = \frac{1}{\mathcal{N}} \prod_\alpha d\xi^*_\alpha d\xi_\alpha$$
$$\mathcal{N} = \begin{cases} 2\pi i & \text{Bosons} \\ 1 & \text{Fermions} \end{cases}$$
$$s = \begin{cases} 1 & \text{Bosons} \\ -1 & \text{Fermions} \end{cases}$$

A.2.2 Complex Scalar $G\Sigma$ Action and SD Equations

There are some tricky points for complex scalar fields. Due to the reason of complex Gaussian variable integration, in order to obtain results consistent with the Schwinger-Dyson equations, the identity operator **1** inserted in the path integral needs to be modified. For details, refer to[34] . Compared with the insertion of **1** in Sarosi, where the coefficient is $\frac{N}{2}$, here it needs to be changed to the coefficient N. Unlike real scalar fields, the functional integral contribution coefficient of the complex scalar field $\overline{\Phi}(\# - \Sigma)\Phi$ is N instead of $\frac{N}{2}$, so N is the matching coefficient.

$$\begin{aligned} 1 &= \int \mathcal{D}G \delta\left(G(\tau,\tau') - \frac{1}{N}\sum \phi^\dagger(\tau)\phi(\tau')\right) \\ &= \int \mathcal{D}G\mathcal{D}\Sigma \exp\left[-N \int d\tau d\tau' \Sigma(\tau,\tau') \left(G(\tau,\tau') - \frac{1}{N}\sum \phi^\dagger(\tau)\phi(\tau')\right)\right] \end{aligned} \tag{A-14}$$

First Set of SD Equations

$$\begin{aligned} &\iint \mathcal{D}\phi^\dagger \mathcal{D}\phi \exp(-\sum \phi^\dagger(\# - \Sigma)\phi - N\Sigma G + \cdots) \\ &= \exp\Big(-N\ln(\det(\# - \Sigma)) - N\Sigma G + \cdots\Big) \end{aligned} \tag{A-15}$$

It can be found that it does not affect the first set of SD equations.

Second Set of SD Equations

Since complex variables are not the focus of this paper, here we only give a general discussion. We first focus on the exponential part of the partition function

$$\exp\left(- N \iint \Sigma G + \text{Gaussian Part}\right) \tag{A-16}$$

Referring to the result of complex Gaussian variables integration: $\propto \sigma^2(XY + YX)$, where X, Y are field quantity terms coupled with $J_{\dots}, \bar{J}_{\dots}$. If we properly arrange the time parameters of the double integral, we will obtain $\sim 2\sigma^2(XY) = \langle J_{\dots}\bar{J}_{\dots}\rangle(XY)$. Therefore, the final SD equation obtained is $\Sigma = \langle J_{\dots}\bar{J}_{\dots}\rangle$(Green Functions), instead of $2\langle J_{\dots}J_{\dots}\rangle$ for real scalar case. This is a significant difference.

A.3 Conformal Dynamics

Rewriting the Schwinger-Dyson (SD) equations in frequency space, we obtain

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \tag{A-17}$$

In the low-energy region where $\omega \ll J$, we can neglect the $-i\omega$ term, leading to a homogeneous set of first-order SD equations. Transforming back to the time domain, we derive the SD equations for the low-energy (deep IR) region:

$$\int d\tau'' G(\tau, \tau'')\Sigma(\tau'', \tau') = -\delta(\tau - \tau') \tag{A-18}$$

$$\Sigma(\tau, \tau') = J^2 G(\tau, \tau')^{q-1}. \tag{A-19}$$

These equations exhibit evident reparameterization invariance and can be considered as CFT_1.

Proof

We perform the following reparameterization transformation, with $\boxed{\Delta = \frac{1}{q}}$, and it is evident that the second set of SD equations remains valid.

$$G(\tau, \tau') \mapsto [\phi'(\tau)\phi'(\tau')]^{\Delta}\, G(\phi(\tau), \phi(\tau')), \tag{A-20}$$

$$\Sigma(\tau, \tau') \mapsto [\phi'(\tau)\phi'(\tau')]^{\Delta(q-1)}\, \Sigma(\phi(\tau), \phi(\tau')). \tag{A-21}$$

For the first set of SD equations, it can also be proven that they still hold.

$$\int d\tau'' \, [\phi'(\tau)\phi'(\tau'')]^{\frac{1}{q}} G(\phi(\tau),\phi(\tau'')) \, [\phi'(\tau'')\phi'(\tau')]^{1-\frac{1}{q}} \, \Sigma(\phi(\tau''),\phi(\tau')) \tag{A-22}$$

$$= \int d\tilde{\phi} G(\phi(\tau),\tilde{\phi})\Sigma(\tilde{\phi},\phi(\tau'))\phi'(\tau') \left[\frac{\phi'(\tau)}{\phi'(\tau')}\right]^{\frac{1}{q}} \tag{A-23}$$

$$= -\phi'(\tau')\delta(\phi(\tau)-\phi(\tau')) \tag{A-24}$$

$$= -\delta(\tau-\tau'). \tag{A-25}$$

Due to the presence of $\delta(\phi(\tau)-\phi(\tau'))$, it is required that $\phi(\tau) \equiv \phi(\tau')$, hence $\frac{\phi'(\tau)}{\phi'(\tau')} \equiv 1$. From this, we find that the system of equations, as described by the equations of motion (EOM), indeed possesses reparameterization invariance. Moreover, we consider this physical system to have time translation invariance, thus $G(\tau_1,\tau_2) \to G(\tau_{12})$.

Therefore, we can use CFT_1 to study it. Based on the discovered reparameterization pattern, we can propose the Ansatz:

$$G_{\text{conformal}}(\tau) = \frac{b}{|\tau|^{2\Delta}}\mathrm{sgn}(\tau). \tag{A-26}$$

The function $\mathrm{sgn}(\tau)$ describes the anti-commutation relations of fermions, and b can be obtained by solving the Schwinger-Dyson (SD) equations. For $w > 0$, the Fourier transform is given by:

$$\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\frac{\mathrm{sgn}\tau}{|\tau|^{2\Delta}} = 2i\mathrm{Im}\left[\int_0^{\infty} d\tau e^{i\omega\tau}\tau^{-2\Delta}\right] \tag{A-27}$$

$$= 2i\mathrm{Im}\left[\left(\frac{i}{\omega}\right)^{1-2\Delta}\Gamma(1-2\Delta)\right] \tag{A-28}$$

$$= 2i\cos(\pi\Delta)\Gamma(1-2\Delta)\frac{1}{\omega^{1-2\Delta}}. \tag{A-29}$$

From the first step to the second step, it is required that $w > 0$. However, from the first equation, we know that $\mathcal{F}[g(w)] = -\mathcal{F}[g(-w)]$, so it can be extended to the case where $w < 0$[14]. Finally, substituting into the solution yields

$$b^q = \frac{1}{\pi J^2}\left(\frac{1}{2}-\frac{1}{q}\right)\tan\frac{\pi}{q}. \tag{A-30}$$

Appendix B More On Determinant Functional

It is known that

$$\ln\det(A) = \mathrm{Tr}\ln(A) \tag{B-1}$$

In chapter 2 we discussed deriving the SD function through rewriting $\ln\det(A) \Rightarrow \mathrm{Tr}\ln(A)$. It is convenient, though hard to process in the coupled SYK case. Here, we are going to deal with the functional determinant using another method.

B.1 Numerical and SD Equations

Before going to discussing the new method, it is worth noticing the advantage of deriving SD eqn through $\mathrm{Tr}\ln(A)$ in certain aspects. According to the discussion with Zheng Jie Ping, using Tr ln for solving directly yields the first set of coupled SYK SD equations in the following form:

The action and the matrix M expansion in the determinant are given by:

$$\frac{I}{N} = -\frac{1}{2}\log\det(M) \tag{B-2}$$

$$+\frac{1}{2}\sum_{ab}\iint \left[\Sigma_{ab}(\tau,\tau')G_{ab}(\tau,\tau')\right. \tag{B-3}$$

$$\left.+\frac{1}{q}t_{ab}J^2 s_{ab}G_{ab}^q(\tau,\tau')\right] d\tau d\tau' + \frac{i\mu}{2}\int \left[G_{LR}(\tau,\tau) - G_{RL}(\tau,\tau)\right] d\tau \tag{B-4}$$

$$M = \delta_{ab}\partial_\tau - \Sigma_{ab} \tag{B-5}$$

$$= \begin{pmatrix} \partial_\tau - \Sigma_{aa} & -\Sigma_{ab} \\ -\Sigma_{ba} & \partial_\tau - \Sigma_{bb} \end{pmatrix} \tag{B-6}$$

$$-\frac{1}{2}\ln\det(M) = -\frac{1}{2}\ln\left((\partial_\tau - \Sigma_{aa})\cdot(\partial_\tau - \Sigma_{bb}) - (-\Sigma_{ab})\cdot(-\Sigma_{ba})\right) \tag{B-7}$$

The variation result is as follows:

$$I_{\delta\Sigma_{aa}} \Rightarrow \tag{B-8}$$

$$\frac{1}{2}\delta\Sigma_{aa} \times \left(\frac{(\partial_\tau - \Sigma_{bb})}{(\partial_\tau - \Sigma_{aa}) \cdot (\partial_\tau - \Sigma_{bb}) - (-\Sigma_{ab}) \cdot (-\Sigma_{ba})} + G_{aa} \right) \tag{B-9}$$

For this type of action, the same denominator for Σ_{ab} is beneficial for numerical solutions of the SD equations; otherwise, divergence may occur. However, for another form of the SD equations, it is better not to use Tr ln for derivation, but rather to directly expand ln det.

B.2 Mathematical Foundations

Lemma 1:

$$\delta \det(A) = \delta A_{ij}\tilde{A}_{ij} \tag{B-10}$$

Proof:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B-11}$$

$$A + \delta A = \begin{pmatrix} a + \delta a & b + \delta b \\ c + \delta c & d + \delta d \end{pmatrix} \tag{B-12}$$

Describing the first-order approximation of the transformation of det:

$$\det(A + \delta A) - \det(A) = \delta a \times d + \delta d \times a - \delta b \times c - \delta c \times b \tag{B-13}$$

The variation of each matrix element can be seen as:

$$\begin{pmatrix} {\color{blue}a} + {\color{red}\delta a} & b + \delta b \\ c + \delta c & {\color{red}d} + {\color{blue}\delta d} \end{pmatrix} \tag{B-14}$$

Defining the negative sign $\tilde{A}_{ij}$ as the algebraic cofactor of the matrix element A_{ij} (note that it is the algebraic cofactor, not just the minor), we can see from the above calculation that $\delta \det(A)_{a_{ij}} = \tilde{A}_{ij}$, which means $\delta \det(A) = \sum_{ij} \delta A_{ij}\tilde{A}_{ij}$.

Lemma 2:

The inverse of a matrix:

$$A^{-1} = \frac{\tilde{A}^T}{\det(A)} \tag{B-15}$$

where $\tilde{A}$ represents the matrix of algebraic cofactors.

- Corollary:

$$\frac{\tilde{A}}{\det(A)} = (A^{-1})^T \tag{B-16}$$

$$\boxed{\delta\det(A) = \delta A_{ij}\tilde{A}_{ij} \quad , \quad A^{-1} = \frac{\tilde{A}^T}{\det(A)}} \tag{B-17}$$

B.3 The First Set of SD Equations for Single SYK

Here, we focus on how to handle the variation of $-\frac{1}{2}\ln\det(\partial - \Sigma)$, while the other parts containing Σ are $\frac{1}{2}\iint \Sigma(\tau_1, \tau_2)G(\tau_1, \tau_2)$.

Formally, it is written as:

$$-\ln\det(A) \quad A = \delta(\tau_1, \tau_2)\partial_\tau - \Sigma \tag{B-18}$$

Varying Σ is equivalent to formally varying A and introducing a negative sign.

$$= -\frac{\tilde{A}}{\det(A)} \times (-1) \tag{B-19}$$

$$= \frac{\tilde{A}}{\det(A)} \tag{B-20}$$

$$= (A^{-1})^T \tag{B-21}$$

$$= (\partial_\tau - \Sigma)^{-1}(\tau_2, \tau_1) \tag{B-22}$$

Remark: We discuss the basis of the matrix as τ, with the row and column indices being τ_1, τ_2. Thus, the effect of transposition is merely to swap the positions of τ_2, τ_1.

Therefore, our variation result gives us:

$$(A^{-1})^T + G = 0 \tag{B-23}$$

$$(\partial_\tau - \Sigma)(\tau_2, \tau_1) + G(\tau_1, \tau_2) = 0 \tag{B-24}$$

Symmetry of $G_{ab}(\tau_1, \tau_2)$:

Although we discuss G_{ab}, for single SYK, we only need to set $a = b$.

First, it is important to note that we are discussing the on-shell G, which should be represented as:

$$G_{ab}(\tau, \tau') = \langle \mathcal{T}\psi_a(\tau)\psi_b(\tau')\rangle \tag{B-25}$$

We must consider the on-shell representation as the time-ordered Green's function, rather than a simple bilinear form, because this relationship ensures the symmetry of $G_{ab}(\tau,\tau')$.

Review of QFT Not Grassmann Numbers but Operators

Although ψ is a Grassmann number, we should consider its operator here. When quantizing fermions, the quantization condition is $\{\psi_i(t),\psi_j(t)\}=\delta_{ij}$. Note that this tells us that for different ij, we have anticommutation relations. However, for the same ij, the situation is different from the unquantized case. In our discussion of the on-shell SD equations, we assume that the quantization condition has already been introduced.

Limitations of Equal-Time Commutation Relations

The limitation is that it is not possible to discuss the commutation relations of fermions at arbitrary times $\psi_i(t) = e^{-Ht}\psi_i(0)e^{iHt}$, so $\{\psi_i(t'),\psi_j(t)\} = (e^{-Ht}\psi_i e^{iH(t-t')}\psi_j e^{iHt'} + (i\leftrightarrow j))$ becomes very complicated. Thus, theoretically, this situation cannot be directly discussed.

Time Ordering Solves the Problem

However, things become much better when we introduce time ordering. For a 1D Majorana system,

$$\mathcal{T}\{\psi(x)\psi(y)\} = \begin{cases}\psi(x)\psi(y), & x>y\\ -\psi(y)\psi(x), & x<y\end{cases} \tag{B-26}$$

Thus, we can see that

$$\mathcal{T}\{\psi(x)\psi(y)\} = -\mathcal{T}\{\psi(y)\psi(x)\}. \tag{B-27}$$

Therefore, we obtain

$$\begin{aligned}G_{ab}(\tau,\tau')^T &= \langle\mathcal{T}\{\psi_a(\tau)\psi_b(\tau')\}\rangle^T\\ &= \langle\mathcal{T}\{\psi_b(\tau')\psi_a(\tau)\}\rangle\\ &= -\langle\mathcal{T}\{\psi_a(\tau)\psi_b(\tau')\}\rangle\\ &= -G_{ab}(\tau,\tau').\end{aligned} \tag{B-28}$$

Thus, the SD equation becomes

$$\begin{aligned}A^{-1}+G^T &= 0,\\ A^{-1} &= G.\end{aligned} \tag{B-29}$$

Appendix C Verification Of Symmetry

$SL(2)$ symmetry for Schwarzian

```
ClearAll["Global`*"]
r[u_] := (a*t[u] + b)/(c*t[u] + d)
Sch[u_] := (2 r'[u] r'''[u] - 3 r''[u]^2)/(2 r'[u]^2)
Sch[u] // Simplify // Expand
```

Composition Law For Schwarzian

```
func[t_] := f[g[t]]
Sch[t_] := (2 D[func[t], t] D[func[t], {t, 3}] -
    3 D[func[t], {t, 2}]^2)/(2 D[func[t], t]^2)
Sch[t] // Expand
```

$SL(2)$ For Propagator

```
    r[u_] := (a*t[u] + b)/(c*t[u] + d)
r'[u1] r'[u2]/(r[u1] - r[u2])^2 // Simplify
```

$SL(2)$ For sin propagator

Discussed in 2 that $SL(2,\mathbb{R})$ transformation of Schwarzian is given by changing $f(u)$ in $\mathrm{Sch}(f(u),u)$ to $\frac{at(u)+b}{ct(u)+d}$. Verified in C. However, we need to ask $ad-bc=1$ to satisfy integration measure in Schwarzian action to be invariant. Therefore, when the action is written in global time, $\mathrm{Sch}(\tan(\frac{t(u)}{2}),u)$, corresponding $SL(2)$ charge originated from the transformation in $\tan(\frac{t(u)}{2})$. To simplify the derivation of how interaction part is $SL(2)$ invariant, we'll simply rescale the $t(u)\rightarrow 2t(u)$.

Following derivation is supported by Jie Ping Zheng

We'll denote $\tan t = f$ for convenience. with $t=(t_1)$ and $\tan t = f = \frac{a\tan u+b}{c\tan u+d}$, with $(ad-bc=1)$

We can see that

$$f' = \tan(t)' = \frac{t'}{\cos^2(t)} = (1+\tan^2(t))t' = (1+f^2)t'$$

Therefore we have

$$\frac{t_1' t_2'}{\sin(t_1 - t_2)^2} = \frac{f_1' f_2' \cos^2 t_1 \cos^2 t_2}{(\sin t_1 \cos t_2 - \cos t_1 \sin t_2)^2} = \frac{f_1' f_2'}{(\tan t_1 - \tan t_2)^2}$$

On the other hand, since $f = \frac{a \tan u + b}{c \tan u + d}$, we have

$$f' = \frac{1}{\cos^2 u} \frac{1}{(c \tan u + d)^2}$$

therefore

$$\frac{t_1'^{\Delta} t_2'^{\Delta}}{\sin(t_1 - t_2)^{2\Delta}} = \frac{f_1'^{\Delta} f_2'^{\Delta} \cos^{2\Delta} t_1 \cos^{2\Delta} t_2}{(\sin t_1 \cos t_2 - \cos t_1 \sin t_2)^{2\Delta}} = \frac{f_1'^{\Delta} f_2'^{\Delta}}{(\tan t_1 - \tan t_2)^{2\Delta}}$$

$$= \frac{1}{\cos^{2\Delta} u_1 \cos^{2\Delta} u_2} \frac{1}{(c \tan u_1 + d)^{2\Delta} (c \tan u_2 + d)^{2\Delta}} \frac{(a \tan u_1 + b - c \tan u_1 + d)}{(a \tan u_1 + b - c \tan u_2 + d)^{2\Delta}}$$

$$= \frac{1}{\cos^{2\Delta} u_1 \cos^{2\Delta} u_2} \frac{1}{[(a \tan u_1 + b)(c \tan u_2 + d) - (a \tan u_2 + b)(c \tan u_1 + d)]^{2\Delta}} \frac{1}{\sin(u_1 - u_2)^{2\Delta}}$$

$$= \frac{1}{\cos^{2\Delta} u_1 \cos^{2\Delta} u_2} \frac{1}{(\sin u_1 \cos u_2 - \cos u_1 \sin u_2)^{2\Delta}} = \frac{1}{(\sin u_1 \cos u_2 - \cos u_1 \sin u_2)^{2\Delta}} \frac{1}{\sin(u_1 - u_2)^{2\Delta}}$$

proving the invariance of the conformal solution under $SL(2, \mathbb{R})$ transformation.